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QUANTUM SYMMETRIC PAIRS AND REPRESENTATIONS OF DOUBLE AFFINE HECKE ALGEBRAS OF TYPE $C^\vee C_n$

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ABSTRACT. We build representations of the affine and double affine braid groups and Hecke algebras of type $C^\vee C_n$, based upon the theory of quantum symmetric pairs (\mathbf{U}, \mathbf{B}) . In the case $\mathbf{U} = \mathcal{U}_q(\mathfrak{gl}_N)$, our constructions provide a quantization of the representations constructed by Etingof, Freund and Ma in [EFM], and also a type $C^\vee C_n$ generalization of the results in [J].

1. INTRODUCTION

In [Ch], Ivan Cherednik introduced the double affine Hecke algebra (abbreviated DAHA, also known as the Cherednik algebra), as a generalization of the affine Hecke algebra (AHA) associated to an affine root system. The DAHA is a quotient of the group algebra of the double affine braid group by additional Hecke relations. Cherednik used these algebras to prove Macdonald's constant term conjecture for Macdonald polynomials. In [S], Sahi constructed a six-parameter DAHA associated to the root system $C^\vee C_n$, and used it to analyze the non-symmetric Macdonald and Koornwinder polynomials.

The degenerate affine Hecke algebra (dAHA) of a Coxeter group was defined by Drinfeld and Lusztig ([Dri], [Lus]). It is a certain multi-parameter deformation of the smash product of the group algebra of the Coxeter group with the coordinate ring of its reflection representation. The degenerate double affine Hecke algebra (dDAHA) of a root system was introduced by Cherednik (see [Ch]). It is a certain multi-parameter deformation of the smash product of the affine Weyl group with the coordinate ring of its reflection representation. The relationship between these algebras and their non-degenerate counterparts is analogous to that between $\mathcal{U}(\mathfrak{g})$ and $\mathcal{U}_q(\mathfrak{g})$: the former may be recovered from the latter by taking quasi-classical limits with respect to the defining parameters.

Motivated by conformal field theory, Arakawa and Suzuki ([AS]) constructed a functor from the category of Harish-Chandra $\mathcal{U}(\mathfrak{gl}_N)$ -bimodules to the category of representations of the dAHA of type A_n for each $n \geq 1$. This construction was extended to the dDAHA of type A_n by Calaque, Enriquez, and Etingof in [CEE], using the theory of ad-equivariant D -modules on the algebraic group $G = \mathrm{GL}_N$.

In [EFM], these constructions were extended to encompass BC_n root systems. More precisely, they considered the symmetric pair of Lie algebras $(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{gl}_N, \mathfrak{gl}_p \times \mathfrak{gl}_q)^1$ associated to the real symmetric pair $(G, K) = (U(N), U(p) \times U(q))$. For each n , there were constructed functors from the category of Harish-Chandra modules for

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¹all Lie algebras are over \mathbb{C} , and $N = p + q$.

(G, K) to the representations of the dAHA, and from the category of K -equivariant D -modules on G/K to the representations of the dDAHA of type BC_n .

In [J], the constructions of [CEE] were quantized to encompass the theory of quantum groups, and the non-degenerate DAHA's of type A_n . Namely, for a quasi-triangular Hopf algebra \mathbf{U} , an integer $n \geq 1$, and $V \in \mathbf{U}\text{-mod}$, there were constructed functors from the category of \mathbf{U} -modules to the category of representations of the affine braid group, and from the category ad-equivariant quantum $D_{\mathbf{U}}$ -modules to the representations of the double affine braid group. In case the braiding on V satisfies a Hecke relation, the functors take values in representations of the AHA and DAHA, respectively. Moreover it was shown that in the case $\mathbf{U} = \mathcal{U}_q(\mathfrak{sl}_N)$, the quasiclassical limit $q \mapsto 1$ recovers the construction of [CEE].

In this paper, we quantize the constructions of [EFM], by appealing to the theory of quantum symmetric pairs, as pioneered by Letzter [L1, L2], and developed further in [DS, Kol, OS], among others. To a simple Lie algebra \mathfrak{g} and an involution $\theta : \mathfrak{g} \rightarrow \mathfrak{g}$ is associated the (classical) symmetric pair $(\mathfrak{g}, \mathfrak{g}^\theta)$. Here \mathfrak{g}^θ is the subalgebra of \mathfrak{g} whose elements are fixed by θ . The quantum analogue of $\mathcal{U}(\mathfrak{g}^\theta)$ is a left (alternatively, right) coideal subalgebra $\mathbf{B} \subset \mathcal{U}_q(\mathfrak{g})$, which specializes to $\mathcal{U}(\mathfrak{g}^\theta)$ as $q \rightarrow 1$. The pair $(\mathcal{U}_q(\mathfrak{g}), \mathbf{B})$ is called a quantum symmetric pair.

For the simple Lie algebras, such pairs were explicitly described by Letzter ([L1, L2]): interestingly, it was shown that in the case of $(\mathfrak{gl}_N, \mathfrak{gl}_p \times \mathfrak{gl}_q)$, there is not a unique quantization, but rather a one-parameter family, $\{\mathbf{B}_\sigma\}_{\sigma \in \mathbb{C}}$, of subalgebras, essentially because the involution θ is replaced by a one-parameter family of automorphisms of $\mathcal{U}_q(\mathfrak{g})$ (see [L1], p. 50). In this case, the algebras \mathbf{B}_σ are known as quantum Grassmannians, and were first introduced by Dijkhuizen, Noumi and Sugitani in the paper [DNS].

Basic algebraic properties of quantum symmetric pairs, and their connection to the so-called reflection equations were established in [KoSt]. In particular, it was explained there how so-called Noumi coideal subalgebras can be constructed canonically, starting from a character of the braided dual, \mathbf{A} , of \mathbf{U} . In the case $\mathbf{U} = \mathcal{U}_q(\mathfrak{gl}_N)$, characters of the reflection equation algebra were classified by Mudrov [Mud], and it was explained in [KoSt] how to extend these to its localization, \mathbf{A} .

Our general setup is as follows. We let \mathbf{U} be a quasitriangular Hopf algebra. We choose a character $f : \mathbf{A} \rightarrow \mathbb{C}$, and denote by $\mathbf{B}_f \subset \mathbf{U}$ the corresponding left Noumi coideal subalgebra. We further choose a character $\chi : \mathbf{B}_f \rightarrow \mathbb{C}$. For each $n \geq 1$, we construct with this data a functor from the category of \mathbf{U} -modules to representations of the affine braid group of type $C^\vee C_n$. Next, we choose a second character $g : \mathbf{A} \rightarrow \mathbb{C}$, and denote by \mathbf{B}'_g the corresponding right Noumi coideal subalgebra. We let $\chi' : \mathbf{B}'_g \rightarrow \mathbb{C}$ be a character. To this data, we associate a functor from the category of $D_{\mathbf{U}}$ -modules (satisfying some technical conditions) to the category to representations of the double affine braid group of type $C^\vee C_n$, by analogy with [EFM]. Our main results are Theorems 5.1, 6.10, 8.1, and 9.1, where we detail the construction of the functors, and apply them in examples to obtain representations of the AHA and DAHA, respectively. We obtain representations of the DAHA with five continuous and one discrete parameter: one parameter for each subalgebra, one parameter for each character, the overall quantization parameter q , and finally the integers N and p defining the classical pair; for the AHA we have three continuous parameters: we choose one subalgebra, its character, and we have the overall quantization parameter q .

The first part of the paper contains the basic constructions, and is organized as follows. In Section 2, we recall the definition of the braid groups and Hecke algebras of type $C^\vee C_n$. In Section 3, we recall the construction of the braided coordinate algebra, and its relation to reflection equations. In Section 4, we recall definitions and notation for quasi-triangular Hopf algebras, Noumi co-ideal subalgebras, and their diagrammatic calculus. In Section 5, we construct representations of the affine braid group using the machinery in the preceding sections. In Section 6, we recall the construction of quantum D -modules and construct representations of double affine braid group from them.

The remainder of the paper is devoted to connections to the AHA and DAHA coming from quantum groups, and is considerably more technical. In Section 7, we recall the quantum group $\mathcal{U}_q(\mathfrak{gl}_N)$, the classical symmetric pair $(\mathfrak{gl}_N, \mathfrak{gl}_p \times \mathfrak{gl}_q)$, and its quantum analog. In Sections 8-9, we show that the constructions of Sections 5 and 6 take values in representations of the AHA and DAHA, respectively, when applied in the context of Section 7. Finally, in Section 10, we compute the quasi-classical limits of our construction and show that they degenerate to those of [EFM].

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2. DOUBLE AFFINE BRAID GROUP AND HECKE ALGEBRA OF TYPE $C^\vee C_n$

2.1. The root system $\Phi^{C^\vee C_n}$ of type $C^\vee C_n$. Let $\mathcal{E}_n = \mathbb{R}^n$, with standard basis ε_i and inner product $(\varepsilon_i, \varepsilon_j) = \delta_{ij}$. We define the set of roots $\Pi^{C^\vee C_n} = \{\pm\varepsilon_i \pm \varepsilon_j\}_{i \neq j} \cup \{\pm\varepsilon_i\} \cup \{\pm 2\varepsilon_i\} \subset \mathcal{E}_n$. Then $\Phi^{C^\vee C_n} := (\mathcal{E}_n, \Pi^{C^\vee C_n})$ defines a non-reduced root system. We choose as a set of positive simple roots:

$$\Pi_+^{C^\vee C_n} = \{\alpha_i = \varepsilon_i - \varepsilon_{i+1}\}_{i=1}^{n-1} \cup \{\alpha_n = \varepsilon_n\}.$$

Let α_0 denote the additional affine positive root. Then $\{\alpha_i, i = 0, \dots, n\}$ form the affine root system of type $C^\vee C_n$. The corresponding affine Dynkin diagram is



For each $\alpha \in \Pi^{C^\vee C_n}$, we s_α denote the corresponding reflection, and let $s_i := s_{\alpha_i}$.

Definition 2.1. The *affine Weyl group*, $\widehat{\mathcal{W}}_n$, of type $C^\vee C_n$, is the group generated by s_0, \dots, s_n , with relations $s_i^2 = 1$, and the braid relations:

$$s_i s_j = s_j s_i, (|i - j| > 1), \quad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, (i \in \{1, \dots, n-1\}),$$

$$s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0 \quad s_{n-1} s_n s_{n-1} s_n = s_n s_{n-1} s_n s_{n-1}.$$

The *Weyl group*, \mathcal{W}_n , of type $C^\vee C_n$, is the subgroup generated by elements s_1, \dots, s_n .

2.2. Double affine braid groups and Hecke algebras in type $C^\vee C_n$.

Definition 2.2. The *affine braid group*, $\widehat{\mathcal{B}}_n$ of type $C^\vee C_n$ is the group generated by T_0, \dots, T_n , subject to the braid relations:

- (1) $T_i T_j = T_j T_i$, ($|i - j| > 1$), $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$, ($i \in \{1, \dots, n-1\}$),
- (2) $T_0 T_1 T_0 T_1 = T_1 T_0 T_1 T_0$ $T_{n-1} T_n T_{n-1} T_n = T_n T_{n-1} T_n T_{n-1}$.

The *braid group*, \mathcal{B}_n , is the subgroup generated by T_1, \dots, T_n .

Definition 2.3. The double affine braid group, $\widetilde{\mathcal{B}}_n$, is the group generated by the affine braid group $\widehat{\mathcal{B}}_n$ and K_0 , subject to the cross relations:

$$\begin{aligned} K_0 T_i &= T_i K_0, \quad (i \in \{2, \dots, n\}); \\ T_1 K_0 T_1 K_0 &= K_0 T_1 K_0 T_1; \\ (3) \quad T_0 T_1^{-1} K_0 T_1 &= T_1^{-1} K_0 T_1 T_0. \end{aligned}$$

Remark 2.4. This presentation for the double affine braid group is different from that in [S] and [EGO], and was chosen to allow the most concise constructions for the current work. It is closely related to presentations in [IS]. In Section 10.6, it is shown that our presentation agrees with the earlier ones.

For later use, we introduce the following notations:

$$\begin{aligned} T_{(i \dots j)} &:= \begin{cases} T_i T_{i+1} \cdots T_{j-1}, & j > i > 0, \\ T_{i-1} \cdots T_{j+1} T_j, & i > j > 0, \\ 1, & i = j. \end{cases} \\ P_i &:= T_i \cdots T_{n-1} T_n T_{n-1} \cdots T_i = T_{(i \dots n)} T_n T_{(n \dots i)}. \end{aligned}$$

Remark 2.5. The group $\widetilde{\mathcal{B}}_n$ admits the following geometric description. Let E be an elliptic curve with coordinate z , and let

$$\widetilde{\text{Conf}}_n(E) := \{(z_1, \dots, z_n) \in E^n \mid z_i \neq \pm z_j, \text{ for } i \neq j, \text{ and } z_i \neq -z_i\}$$

$$\text{Conf}_n(E) := \widetilde{\text{Conf}}_n(E) / ((\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n),$$

where each $\mathbb{Z}/2\mathbb{Z}$ replaces z_i with $-z_i$, and S_n permutes the factors. Then one can check that $\pi_1(\text{Conf}_n(E)) \cong \widetilde{\mathcal{B}}_n$. This is a double affine version of the usual identification [Br] of \mathcal{B}_n with $\pi_1(\mathfrak{h}_{reg}/\mathcal{W}_n)$, where

$$\mathfrak{h}_{reg} := \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid z_i \neq 0, z_i \neq \pm z_j, \text{ for } i \neq j\}.$$

See Section 10.6 for further discussion.

We fix a field \mathcal{K} , and let $v, t, t_0, u_0, t_n, u_n \in \mathcal{K}^\times$.² For an operator X and a parameter x , we use the notation $X \sim x$ to mean that X satisfies the Hecke relation $(X - x)(X + x^{-1}) = 0$.

Definition 2.6. The *double affine Hecke algebra*, $\mathcal{H}_n(v, t, t_0, t_n, u_0, u_n)$, of type $C^\vee C_n$, is the quotient of the group algebra $\mathcal{K}[\widetilde{\mathcal{B}}_n]$ by the Hecke relations:

$$T_0 \sim t_0, \quad T_n \sim t_n, \quad K_0 \sim u_n, \quad (v K_0 P_1 T_0)^{-1} \sim u_0, \quad T_1, \dots, T_{n-1} \sim t.$$

²For historical reasons, it is common to replace these parameters formally with their square roots. For simplicity, we have dropped this convention.

The *affine Hecke algebra*, $\mathcal{H}_n(t, t_0, t_n)$, of type $C^\vee C_n$, is the quotient of the group algebra $\mathcal{K}[\widehat{\mathcal{B}}_n]$ by the relations:

$$T_0 \sim t_0, \quad T_n \sim t_n, \quad T_1, \dots, T_{n-1} \sim t.$$

The *Hecke algebra*, $H_n(t, t_n)$, of type $C^\vee C_n$, is the quotient of the group algebra $\mathcal{K}[\mathcal{B}_n]$ by the relations:

$$T_n \sim t_n, \quad T_1, \dots, T_{n-1} \sim t.$$

Remark 2.7. $\mathcal{H}_n(t, t_0, t_n)$ and $H_n(t, t_n)$ are subalgebras of $\mathcal{H}_n(v, t, t_0, t_n, u_0, u_n)$ in the obvious way.

Remark 2.8. There are three variants of the above setup, depending on the choice of \mathcal{K} . One may consider: $\mathcal{K} = \mathbb{C}$, and the parameters are numerical, $\mathcal{K} = \mathbb{C}(v, t, t_0, t_n, u_0, u_n)$ and the parameters are indeterminates, $\mathcal{K} = \mathbb{C}((\hbar))$ and the parameters are formal Laurent series. The latter will appear most notably in Section 10, and in that case, we also complete all algebras with respect to \hbar .

3. CHARACTERS OF THE BRAIDED DUAL AND THE REFLECTION EQUATION

In this section we recall a categorical construction of a certain quantization of the algebra of functions on an algebraic group, which Majid dubbed the covariantized coordinate algebra, or simply the braided group. For clarity of presentation, we recall some elementary constructions in the theory of tensor categories and phrase our constructions in these terms; of course, we could just as well phrase constructions in terms of generators and relations (see Example 4.5). For details about locally finite tensor categories, see [De1], [De2].

Definition 3.1. An abelian category \mathcal{C} is called *locally finite* if every object $X \in \mathcal{C}$ has finite length, and all Hom spaces are finite dimensional.

Example 3.2. The category of finite dimensional modules over an algebra (possibly infinite dimensional) is a locally finite abelian category, equipped with a functor to vector spaces.

Let $(\mathcal{C}, \otimes, \sigma)$ be a locally finite braided tensor category, and let $\mathcal{C} \boxtimes \mathcal{C}$ denote its Deligne tensor square. If \mathcal{C} is semisimple, then $\mathcal{C} \boxtimes \mathcal{C}$ is also, with simples $X \boxtimes Y$, for $X, Y \in \mathcal{C}$ simple. In any case, we will refer to objects in $\mathcal{C} \boxtimes \mathcal{C}$ of the form $V \boxtimes W$ as *pure* objects: every object in $\mathcal{C} \boxtimes \mathcal{C}$ is a finite iterated extension of pure objects. $\mathcal{C} \boxtimes \mathcal{C}$ is also a tensor category with tensor product \otimes_2 , given on pure objects by:

$$(V \boxtimes W) \otimes_2 (X \boxtimes Y) := (V \otimes X) \boxtimes (W \otimes Y).$$

$\mathcal{C} \boxtimes \mathcal{C}$ becomes a braided tensor category with braiding $\sigma_2 := \sigma \boxtimes \sigma$. The tensor product on \mathcal{C} gives a functor

$$T : \mathcal{C} \boxtimes \mathcal{C} \rightarrow \mathcal{C}, \quad V \boxtimes W \mapsto V \otimes W.$$

We can equip T with the structure of a tensor functor by using the braiding $\sigma_{W,X}$:

$$\beta : T(V \boxtimes W) \otimes T(X \boxtimes Y) = V \otimes W \otimes X \otimes Y \xrightarrow{\sigma_{W,X}} V \otimes X \otimes W \otimes Y = T(V \boxtimes W \otimes_2 X \boxtimes Y).$$

There is an important ind-algebra³ $\mathbf{A} = \text{CoEnd}(\mathcal{C})$ in $\mathcal{C} \boxtimes \mathcal{C}$, first constructed by Majid [Maj]. As we will use it extensively in what follows, we recall its construction

³An ind-object in \mathcal{C} is a direct limit of objects in \mathcal{C} , but not, in general, itself an object of \mathcal{C} . Rather it is an object in a completion of \mathcal{C} with respect to inductive limits; this distinction is not particularly important for us.

here. To begin, we consider the (very large) ind-object \tilde{A} in $\mathcal{C} \boxtimes \mathcal{C}$:

$$\tilde{A} = \bigoplus_{V \in \mathcal{C}} V^* \boxtimes V.$$

Let $Q \subset \tilde{A}$ denote the sum over all V, W , and $\phi : V \rightarrow W$ of the images in \tilde{A} of

$$(4) \quad x_\phi := \phi^* \boxtimes \text{id}_V - \text{id}_W^* \boxtimes \phi \in \text{Hom}(W^* \boxtimes V, V^* \boxtimes V \oplus W^* \boxtimes W).$$

As an ind-object in \mathcal{C} , we define $\mathbf{A} := \tilde{A}/Q$. Note that for any object $V \in \mathcal{C}$, we have a canonical map $i_V : V^* \boxtimes V \rightarrow \mathbf{A}$. A multiplication $\mu : \mathbf{A} \otimes_2 \mathbf{A} \rightarrow \mathbf{A}$ is given on each $V^* \boxtimes V, W^* \boxtimes W$ by

$$\mu : (V^* \otimes W^*) \boxtimes (V \otimes W) \xrightarrow{\sigma_{V^*, W^*} \boxtimes \text{id}} (W^* \otimes V^*) \boxtimes (V \otimes W) \cong (V \otimes W)^* \boxtimes (V \otimes W),$$

which makes \mathbf{A} into a unital associative algebra in $\mathcal{C} \boxtimes \mathcal{C}$ (one uses the braid relations on the first factor). By tensor functoriality, $T(\mathbf{A})$ also becomes a unital associative algebra in \mathcal{C} with multiplication $T(\mu) \circ \beta$. Furthermore, $T(\mathbf{A})$ carries the structure of a coalgebra in \mathcal{C} , with comultiplication defined on generators $V^* \otimes V$:

$$\Delta := \text{id}_V^* \otimes \text{coev}_V \otimes \text{id}_V : V^* \otimes V \rightarrow V^* \otimes V \otimes V^* \otimes V \subset T(\mathbf{A}) \otimes T(\mathbf{A}).$$

The counit is defined on generators by the pairing $\text{ev} : V^* \otimes V \rightarrow \mathbb{1}$. Any object in \mathcal{C} is naturally both a right and left comodule over $T(\mathbf{A})$ via the maps

$$(5) \quad \Delta_V^R := \text{coev}_V \otimes \text{id} : V \rightarrow V \otimes V^* \otimes V \subset V \otimes T(\mathbf{A}),$$

$$(6) \quad \Delta_V^L := \text{id} \otimes \text{coev}_{V^*} : V \rightarrow V \otimes^* V \otimes V \subset T(\mathbf{A}) \otimes V.$$

Finally, we have the antipode map $S : T(\mathbf{A}) \rightarrow T(\mathbf{A})$ defined on generators by

$$S|_{V^* \otimes V} := (u_V \otimes \text{id}) \circ \sigma_{V^*, V} : V^* \otimes V \rightarrow V^{**} \otimes V^*,$$

where $u_V : V \rightarrow V^{**}$ is the Drinfeld element (see, e.g. [KlSch], p. 247). Together these maps make $T(\mathbf{A})$ into a braided Hopf algebra in \mathcal{C} , as defined by Majid [Maj]. Note that $\Delta^L = \sigma_{V, \mathbf{A}} \circ (\text{id} \otimes S) \circ \Delta^R$.

Remark 3.3. A more concise description of \mathbf{A} may be given in the language of module categories. For a \mathcal{C} -module category \mathcal{M} , and $M, N \in \mathcal{M}$, we let $\underline{\text{Hom}}(M, N) \in \mathcal{C}$ denote the representing object for the functor $\text{Hom}_{\mathcal{M}}(\bullet \otimes M, N)$ (called the *inner Homs* from M to N). When $M = N$, $\underline{\text{Hom}}(M, M)$ has a natural algebra structure (see [EO] for details). Any tensor category \mathcal{C} has the structure of a $\mathcal{C} \boxtimes \mathcal{C}^{\otimes-\text{op}}$ module-category, given by $(X \boxtimes Y) \otimes M := X \otimes M \otimes Y$. Thus we have an algebra $\mathbf{A}' := \underline{\text{Hom}}(\mathbb{1}, \mathbb{1}) \in \mathcal{C} \boxtimes \mathcal{C}^{\otimes-\text{op}}$; \mathbf{A}' represents the functor taking $X \boxtimes Y$ to the co-invariants of $X \otimes Y$. Finally \mathbf{A} is the $\mathcal{C} \boxtimes \mathcal{C}$ algebra equivalent to \mathbf{A}' via the functor $\text{id} \boxtimes \sigma : \mathcal{C} \boxtimes \mathcal{C} \rightarrow \mathcal{C} \boxtimes \mathcal{C}^{\otimes-\text{op}}$. We will not use this construction of \mathbf{A} in later sections, but rather its explicit presentation in terms of the relations of equation (4).

Key to applications in Lie theory and quantum groups is the observation that when \mathcal{C} is semi-simple, \mathbf{A} admits the following Peter-Weyl decomposition:

Proposition 3.4. *Suppose that \mathcal{C} is semi-simple. Then we have:*

$$\mathbf{A} \cong \bigoplus_{V \text{ simple}} V^* \boxtimes V,$$

where the sum counts each isomorphism class of simple objects exactly once.

Proof. Apply the relations in equation (4) to isomorphisms $\phi : V \rightarrow W$, to reduce the sum to isomorphism classes of objects V . Apply equation (4) to the projections and inclusions of simple components, to further reduce the sum to the simple objects V . \square

4. QUASI-TRIANGULAR HOPF ALGEBRAS

For the rest of the paper, we work under the assumption that \mathcal{C} is a locally finite braided tensor subcategory of the category of finite dimensional complex representations of a quasi-triangular Hopf algebra \mathbf{U} . We denote by F the corresponding tensor functor to vector spaces. For any \mathbf{U} -module $V \in \mathcal{C}$, we denote the action by $\rho_V : \mathbf{U} \rightarrow \text{End}_{\mathbb{C}}(V)$.

4.1. The universal \mathbf{R} -matrix and \mathbf{L} -operators. Recall (see, e.g. [KLSch] for details) that a quasi-triangular Hopf algebra is a Hopf algebra \mathbf{U} , with an invertible element $\mathcal{R} = \sum_i r_i \otimes r'_i \in \widehat{\mathbf{U} \otimes \mathbf{U}}$ ⁴, called the universal \mathbf{R} -matrix, such that $\Delta^{\text{cop}}(u) = \mathcal{R} \Delta(u) \mathcal{R}^{-1}$, for all $u \in \mathbf{U}$, and

$$(\Delta \otimes \text{id})(\mathcal{R}) = \mathcal{R}_{13} \mathcal{R}_{23}, \quad (\text{id} \otimes \Delta)(\mathcal{R}) = \mathcal{R}_{13} \mathcal{R}_{12},$$

where $\mathcal{R}_{12} = \sum_i r_i \otimes r'_i \otimes 1$, $\mathcal{R}_{13} = \sum_i r_i \otimes 1 \otimes r'_i$, and $\mathcal{R}_{23} = \sum_i 1 \otimes r_i \otimes r'_i$.

The braiding in \mathcal{C} is given by

$$(2) \quad \sigma_{V,W} = \tau_{V,W} \circ R_{V,W} : V \otimes W \xrightarrow{\cong} W \otimes V,$$

for any $V, W \in \mathcal{C}$. Here $R_{V,W} := \rho_V \otimes \rho_W(\mathcal{R})$, $\tau_{V,W}$ is the flip operator $V \otimes W \rightarrow W \otimes V, v \otimes w \mapsto w \otimes v$. We will suppress “ $\otimes \text{id}$ ” from morphisms on tensor products when it is clear from context (e.g. $\sigma_{V,W} := \text{id} \otimes \sigma_{V,W} : \bullet \otimes V \otimes W \rightarrow \bullet \otimes W \otimes V$).

Remark 4.1. Usually \mathcal{R} is assumed to lie in $\mathbf{U} \otimes \mathbf{U}$ rather than its completion. However many examples - in particular those coming from quantum groups - fall into this more general context, so we adopt this definition. One could alternatively work with comodules over co-quasitriangular Hopf algebras, but we prefer the present, equivalent, formalism.

For any \mathbf{U} -module V , we define the “ \mathbf{L} -operators”:

$$\begin{aligned} L_V^+ &= (\text{id} \otimes \rho_V)(\mathcal{R}) \in \mathbf{U} \otimes \text{End}_{\mathbb{C}}(V), \\ L_V^- &= (\rho_V \otimes \text{id})(\mathcal{R}^{-1}) \in \text{End}_{\mathbb{C}}(V) \otimes \mathbf{U}. \end{aligned}$$

For a basis of V , $\{e_i\}$, we define elements $l_{ij}^{V\pm} \in \mathbf{U}$ by

$$(8) \quad L_V^+(1 \otimes e_j) = \sum_i l_{ij}^{V+} \otimes e_i, \quad \text{and} \quad L_V^-(e_j \otimes 1) = \sum_i e_i \otimes l_{ij}^{V-}.$$

We have:

$$(9) \quad \Delta(l_{ij}^{V\pm}) = \sum_k l_{ik}^{V\pm} \otimes l_{kj}^{V\pm}.$$

⁴For an algebra A , let \hat{A} denote its profinite completion, i.e. the completion in the topology in which a basis of neighborhoods of zero is formed by the annihilators of finite dimensional modules. In other words, $\sum_k a_k \in \hat{A}$ if, and only if, for all $V \in A\text{-mod}$ finite dimensional, $a_k V = 0$, for $k \gg 0$.

4.2. The CoEnd algebra \mathbf{A} . A fiber functor on $\mathcal{C} \boxtimes \mathcal{C}$ is defined by $F_2 := F \circ T : \mathcal{C} \boxtimes \mathcal{C} \rightarrow \text{Vect}$. Now let $\mathbf{A} = \text{CoEnd}(\mathcal{C})$ be the ind-algebra in \mathcal{C} defined in Section 3. Then $F_2(\mathbf{A})$ becomes an algebra in the usual sense (i.e. in the category of vector spaces), by tensor functoriality.

Remark 4.2. In this case, it is well known that $F_2(\mathbf{A})$ is isomorphic as a coalgebra to the restricted dual \mathbf{U}° of \mathbf{U} , and that the product in $F_2(\mathbf{A})$ is twisted from that of \mathbf{U}° by a certain cocycle built from the braiding, hence the name “braided dual”.

For any $V \in \mathcal{C}$, recall the comodule maps Δ_V^R, Δ_V^L defined in equations (5), (6). Fixing a basis of V , we can write them as matrices, with coefficients in $F_2(\mathbf{A})$:

$$\Delta_V^R = \sum_{i,j=1}^{\dim V} E_i^j \otimes a_j^i(V), \quad \Delta_V^L = \sum_{i,j=1}^{\dim V} \tilde{a}_j^i(V) \otimes E_i^j.$$

Here E_i^j is the matrix $E_i^j v_k = \delta_{jk} v_i$. Now suppose $V, W \in \mathcal{C}$ with chosen basis. Define

$$\begin{aligned} C_V^R &:= \sum_{i,j=1}^{\dim V} E_i^j \otimes \text{id} \otimes a_j^i(V) \in \text{End}_{\mathcal{C}}(V) \otimes \text{End}_{\mathcal{C}}(W) \otimes T(\mathbf{A}), \\ C_W^R &:= \sum_{i,j=1}^{\dim W} \text{id} \otimes E_k^l \otimes a_l^k(W) \in \text{End}_{\mathcal{C}}(V) \otimes \text{End}_{\mathcal{C}}(W) \otimes T(\mathbf{A}). \end{aligned}$$

Similarly, we have operators C_V^L, C_W^L defined using Δ_V^L, Δ_W^L instead.

Theorem 4.3 ([Maj],[DKM]. See [J], Proposition 2.14 for a short proof.). *For any $V, W \in \mathcal{C}$, the generators $V^* \otimes V$ and $W^* \otimes W$ in $F_2(\mathbf{A})$ satisfy the relations of the reflection equation algebra:*

$$(10) \quad \sigma_{W,V} C_V^R \sigma_{V,W} C_W^R = C_W^R \sigma_{W,V} C_V^R \sigma_{V,W},$$

$$(11) \quad \sigma_{W,V} C_W^L \sigma_{V,W} C_V^L = C_V^L \sigma_{W,V} C_W^L \sigma_{V,W}.$$

Example 4.4. If we take \mathcal{C} to be the symmetric category of finite dimensional $\mathcal{U}(\mathfrak{g})$ -modules, then the resulting algebra $F_2(\mathbf{A})$ is the coordinate algebra $\mathcal{O}(G)$ for the connected, simply connected algebraic group with Lie algebra \mathfrak{g} .

Example 4.5. If we instead take \mathcal{C} to be the category of finite dimensional, type I $\mathcal{U}_q(\mathfrak{gl}_N)$ -modules (see Section 7.1), the resulting algebra $F_2(\mathbf{A})$ is Majid’s covariantized coordinate algebra. $F_2(\mathbf{A})$ is twist equivalent (though not isomorphic) to the usual dual quantum group $\mathcal{O}_q(G)$, and has been suggested ([Maj], [DM1]) as a preferable replacement for $\mathcal{O}_q(G)$ in the context of braided geometry, as it is constructed to be covariant for the coadjoint action of $\mathcal{U}_q(\mathfrak{g})$.

We can write a presentation of $F_2(\mathbf{A})$ explicitly as follows. It is well-known that in this case, \mathcal{C} is generated as a tensor category by the defining representation \mathbb{C}^N with highest weight $(1, 0, \dots, 0)$, together with the dual of the determinant representation $\Lambda_q^N(\mathbb{C}^N)$. It follows immediately that $F_2(\mathbf{A})$ is generated as an algebra by the elements $a_{f,v}, f \in (\mathbb{C}^N)^*, v \in \mathbb{C}^N$, subject to the relations (10) with $V = W = \mathbb{C}^N$, and the inverse of the central element \det_q . Even more explicitly, we can choose the standard basis $\{e_i\}$ of weight vectors for V_0 , and its dual basis

$\{e^i\}$ for V_0^* , and set $a_j^i := a_{e^i, e_j}$. Then $F_2(\mathbf{A})$ is the algebra generated by the a_j^i and \det_q^{-1} , subject to relations:

$$(12) \quad \sum R_{ms}^{ik} a_l^s R_{un}^{lm} a_v^n = \sum a_l^i R_{mn}^{lk} a_s^n R_{uv}^{sm}.$$

As has been noted in many places, these are precisely the so-called “reflection equations”.

4.3. Characters of $F_2(\mathbf{A})$. Now suppose that $f : F_2(\mathbf{A}) \rightarrow \mathbb{C}$ is a character (homomorphism of algebras). For $V \in \mathcal{C}$, let $J_V := \sum_{i,j} f(a_j^i(V)) E_i^j$, and $J'_V := \sum_{i,j} f(\tilde{a}_j^i(V)) E_i^j$. Then we have the following well-known

Proposition 4.6. *For all $V, W \in \mathcal{C}$, we have the following relation in $\text{End}_{\mathbb{C}}(V \otimes W)$:*

$$(13) \quad \sigma_{W,V} J_V \sigma_{V,W} J_W = J_W \sigma_{W,V} J_V \sigma_{V,W}.$$

$$(14) \quad \sigma_{W,V} J'_W \sigma_{V,W} J'_V = J'_V \sigma_{W,V} J'_W \sigma_{V,W}.$$

Proof. Apply f to the equations (10) and (11). □

We will refer to equations (13) and (14) as the “right-handed” and “left-handed” reflection equations, respectively.

4.4. Coideal subalgebras associated to characters. The operators J_V and J'_V constructed from f in the previous section are not, in general, realized as morphisms of \mathbf{U} -modules. Rather, they are morphisms of \mathbf{B}_f -modules (resp. \mathbf{B}'_f -modules), for certain coideal subalgebras $\mathbf{B}_f, \mathbf{B}'_f \subset \mathbf{U}$ constructed in [KoSt], which we now recall. Let \mathbf{B}_f and \mathbf{B}'_f denote the subalgebras of \mathbf{U} generated by the sets:

$$(15) \quad \Phi_f := \{c_{il} = \sum_{j,k=1}^N l_{ij}^{V+}(J_V)_{jk} S(l_{kl}^{V-}) | i, l = 1, \dots, N\},$$

$$(16) \quad \Phi'_f := \{c'_{il} = \sum_{j,k=1}^N S(l_{ij}^{V-})(J'_V)_{jk} l_{kl}^{V+} | i, l = 1, \dots, N\},$$

respectively. Here S is the antipode of the Hopf algebra \mathbf{U} and $N = \dim V$. \mathbf{B}_f and \mathbf{B}'_f are independent on the choice of basis, and it follows from (9) that they form left and right coideal subalgebras, respectively:

$$\Delta(\mathbf{B}_f) \subset \mathbf{U} \otimes \mathbf{B}_f, \quad \Delta(\mathbf{B}'_f) \subset \mathbf{B}'_f \otimes \mathbf{U}.$$

Proposition 4.7. *The operator $J_V \in \text{End}_{\mathbb{C}}(V)$ is \mathbf{B}_f -linear: $J_V(xv) = xJ_V(v)$ for all $v \in V$ and $x \in \mathbf{B}_f$. The operator $J'_V \in \text{End}_{\mathbb{C}}(V)$ is \mathbf{B}'_f -linear: $J'_V(xv) = xJ'_V(v)$ for all $v \in V$ and $x \in \mathbf{B}'_f$.*

Proof. Similar proofs have appeared in many sources, e.g. [KoSt], [DS], [NS]; we include a proof here for the reader’s convenience. We prove the statement for J_V ; the statement for J'_V is similar. To show that J_V commutes with all the $\rho_V(c_{il})$ is equivalent to showing that $(\text{id} \otimes J_{V_2})$ commutes with $x = \sum E_i^l \otimes \rho_V(c_{il}) \in \text{End}_{\mathbb{C}}(V_1 \otimes V_2)$, where $V_1 = V_2 = V$. We observe that

$$x = \sum E_i^l \otimes \rho_V(l_{ij}^{V+}(J_V)_{jk} S(l_{kl}^{V-})) = \sigma_{V_2, V_1} J_{V_1} \sigma_{V_1, V_2},$$

so that the claim reduces to the right handed reflection equation. □

Remark 4.8. The proof of Proposition 4.7 relies on the observation that the matrix coefficients of $\sigma_{V_2, V_1} J_{V_1} \sigma_{V_1, V_2}$ are precisely the generators of \mathbf{B}_f . The same observation provides the key steps in Lemmas 6.9 and 9.4.

4.5. J_V -decorated Tangle Diagrams in $\mathcal{C} \boxtimes \mathcal{C}$. Morphisms in a braided tensor category may be conveniently manipulated using tangle diagram notation (see, e.g. [K], Chapter XIV). It will be necessary to extend the tangle diagram notation in two ways: first, we consider morphisms in the Deligne tensor product $\mathcal{C} \boxtimes \mathcal{C}$; second, we admit morphisms J_V and J'_V which are not morphisms in \mathcal{C} but rather in the \mathcal{C} -module categories of representations of the coideal subalgebras \mathbf{B}_f and \mathbf{B}'_f from Section 4.4.

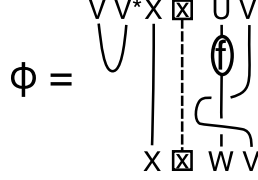


FIGURE 1. The tangle diagram for the morphism ϕ of equation (17).

To depict an object of $\mathcal{C} \boxtimes \mathcal{C}$, we draw the objects alongside one another, separated by the \boxtimes symbol. For a morphism $f \boxtimes g$ in $\mathcal{C} \boxtimes \mathcal{C}$, we draw the corresponding tangle diagrams alongside one another, joining the \boxtimes symbols with a dotted line. We follow the convention from [K] that morphisms move up the page. For example, for $f \in \text{Hom}(W, U)$, Figure 1 depicts the morphism:

$$(17) \quad \phi = (\text{coev}_V \otimes \text{id}_X) \boxtimes ((f \otimes \text{id}_V) \circ \sigma_{W, V}^{-1} \circ \sigma_{V, W}^{-1}).$$

The linear maps J_V and J'_V do not commute with the braiding in the ordinary way, but may instead be manipulated in a tangle diagram by applying equations (13) and (14), as depicted in Figure 2.

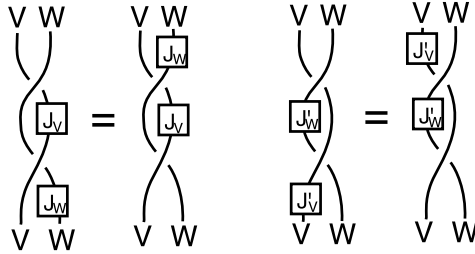


FIGURE 2. Equality of J -decorated tangle diagrams representing equations (13) and (14), respectively.

5. SOME NEW REPRESENTATIONS OF THE AFFINE BRAID GROUP OF TYPE $C^\vee C_n$

Let \mathcal{C} , F , and f be as in Section 4. For any objects $M, V_1, \dots, V_n \in \mathcal{C}$, consider the vector space⁵:

$$F_{V_1, \dots, V_n}^f(M) := M \otimes V_1 \otimes \dots \otimes V_n.$$

For simplicity we will take $V_1 = \dots = V_n = V$ (though it is still convenient to retain the indices), and in this case abbreviate $F_{V,n}^f := F_{V_1, \dots, V_n}^f$. Our goal in this section is to construct an action of $\widehat{\mathcal{B}}_n$ on $F_{n,V}^f(M)$. Recall that the character f determines a map $J_{V_i} : V_i \rightarrow V_i$, for each i .

In the following construction, we make frequent use of the maps J_V . As was mentioned in Section 4.5, the only flexibility in moving the morphisms J_V about a tangle comes from the reflection equation for J_V , and so we make repeated use of that identity throughout. We will use the abbreviation QYBE (quantum Yang-Baxter equation) to refer to relations of undecorated tangle diagrams.

5.1. The action of \mathcal{B}_n . Let $T_i = \sigma_{V_i, V_{i+1}}$, for $i = 1, \dots, n-1$. Then it is well known that the T_i 's satisfy the braid relations (1). Now let $T_n = J_{V_n} = \text{id}_M \otimes \text{id}^{\otimes(n-1)} \otimes J_V$. Then the required relation

$$T_n T_{n-1} T_n T_{n-1} = T_{n-1} T_n T_{n-1} T_n$$

is equivalent to the right-handed reflection equation for J_{V_n} . Thus the above construction gives an action of \mathcal{B}_n on $F_{n,V}^f(M)$. Related constructions have appeared in [KoSt, tD, tDHO], under the name “universal cylinder forms”.

5.2. The action of T_0 . We let

$$T_0 = P_1^{-1}(\sigma_{V_1, M} \circ \sigma_{M, V_1})^{-1}$$

See Figure 3 for the tangle diagram associated to T_0 . It is straightforward to verify that $T_i T_0 = T_0 T_i$ for $i \geq 2$. We check $T_1 T_0 T_1 T_0 = T_0 T_1 T_0 T_1$ in Figure 4.

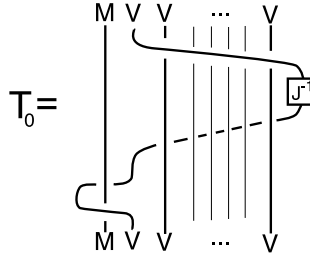


FIGURE 3. The morphism T_0

We have proven the following:

Theorem 5.1. *The operators T_0, \dots, T_n define a representation of $\widehat{\mathcal{B}}_n$ on $F_{n,V}^f(M)$. Thus we have a functor:*

$$F_{n,V}^f : \mathcal{C} \rightarrow \widehat{\mathcal{B}}_n\text{-mod}, \quad M \mapsto F_{n,V}^f(M).$$

⁵While f does not affect the underlying vector space, it impacts the functor constructed in Theorem 5.1, and so we introduce the notation here.

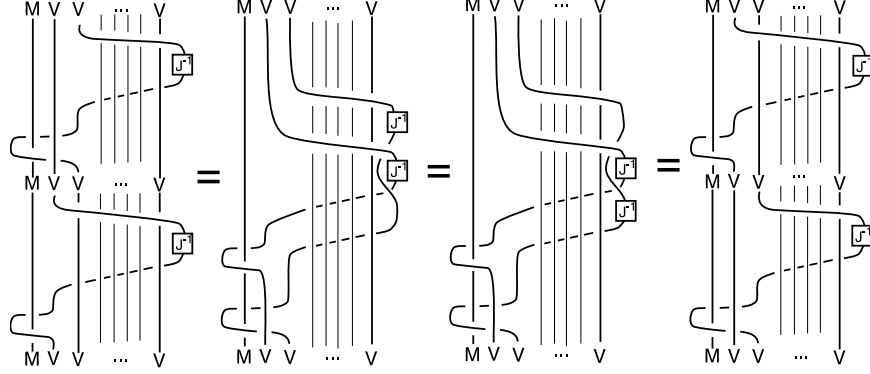


FIGURE 4. Proof of relation $T_1T_0T_1T_0 = T_0T_1T_0T_1$. The first and third equalities use only QYBE, while the second uses the reflection equation for J .

Remark 5.2. The *pure* (double, affine) braid group on n strands is the kernel of the natural projection from the (double, affine) braid group to the symmetric group \mathcal{S}_n . It is clear from the proof that Theorem 5.1 extends more generally to the pure affine braid group on n strands, if we drop the assumption that all V_i are equal. Alternatively, given V_1, \dots, V_n possibly distinct, we can construct a similar action of the full affine braid group on the sum

$$\tilde{F}_{V_1, \dots, V_n}^f := \bigoplus_{\sigma \in \mathcal{S}_n} M \otimes V_{\sigma(1)} \otimes \dots \otimes V_{\sigma(n)}.$$

The same remark applies to Theorem 6.10.

6. SOME NEW REPRESENTATIONS OF THE DOUBLE AFFINE BRAID GROUP OF TYPE $C^\vee C_n$

6.1. Quantum D -modules. Let \mathbf{U} be a quasi-triangular Hopf algebra, and \mathcal{C} be a locally finite braided tensor subcategory of $\mathbf{U}\text{-mod}$, as in Section 4. The algebra $D_{\mathbf{U}}$ of quantum differential operators⁶ is a Hopf algebra analog of the algebra of differential operators on the algebraic group G with Lie algebra \mathfrak{g} : when $\mathbf{U} = \mathcal{U}(\mathfrak{g})$, we have $D_{\mathbf{U}} = D(G)$. In this section, we recall the definition of $D_{\mathbf{U}}$, and some constructions from [VV] involving it. We have followed their notation as closely as possible, though there are a few differences (in particular, see Remark 6.5).

Let \mathbf{A} be the braided dual algebra defined in Section 3, and let $\tilde{\mathbf{A}} = (F \boxtimes F)(\mathbf{A})$. That is, we regard \mathbf{A} as an algebra in vector spaces, where it is equipped with a $(\mathbf{U} \otimes \mathbf{U})$ -action. Note that $\tilde{\mathbf{A}}$ is not isomorphic to $F_2(\mathbf{A})$ considered previously (although it is obviously twist equivalent). $\tilde{\mathbf{A}}$ is also a twist-equivalent to a subalgebra of \mathbf{U}^* . Namely, each $f \boxtimes v \in \tilde{\mathbf{A}}$, gives a linear functional $a_{f,v} : \mathcal{U} \rightarrow \mathbb{C}$ by $a_{f,v}(u) = f(uv)$. The product in $\tilde{\mathbf{A}}$ is such that

$$(18) \quad (a_{f,v} a_{g,w})(u) = \sum_i a_{r'_i g \otimes r_i f, v \otimes w}(u) = \sum_i a_{r_i f, v}(u_{(1)}) a_{r'_i g, w}(u_{(2)}).$$

⁶ \mathbf{A} and thus $D_{\mathbf{U}}$ depend on the choice of \mathcal{C} , but we will suppress this in the notation

We use “ \triangleright ” to denote the left adjoint action of \mathbf{U} on itself: for $x, y \in \mathbf{U}$, $y \triangleright x := y_{(1)}xS(y_{(2)})$, where $\Delta(x) = x_{(1)} \otimes x_{(2)}$ is Sweedler’s implicit sum notation for the coproduct. As there is no risk of confusion, we use the same symbol to denote the action of $\mathbf{U} \otimes \mathbf{U}$ on $\tilde{\mathbf{A}}$: for $x, y \in \mathbf{U}$, and $f \otimes v \in \tilde{\mathbf{A}}$, we let $(x \otimes y) \triangleright (f \otimes v) := xf \otimes yv$. In particular, the coadjoint action of $u \in \mathbf{U}$ on $a \in \tilde{\mathbf{A}}$ is given by $\Delta(u) \triangleright a$. Recall that for vector spaces V, W , $\tau_{V,W} : V \otimes W \rightarrow W \otimes V$ denotes the tensor flip.

Let \mathbf{U}' denote the left coideal subalgebra in \mathbf{U} consisting of elements x which generate a finite dimensional submodule under the adjoint action.

Definition 6.1. The algebra $D_{\mathbf{U}}$ of quantum differential operators has underlying vector space $\tilde{\mathbf{A}} \otimes \mathbf{U}$; the natural inclusions of $\tilde{\mathbf{A}} \otimes 1$ and $1 \otimes \mathbf{U}$ are algebra homomorphisms, and the commutation relations are given by the smash product:

$$(1 \otimes x)(a \otimes 1) = \sum_{i,j} ((x_{(1)} \otimes 1) \triangleright a) \otimes x_{(2)}, \text{ for } a \in \tilde{\mathbf{A}}, x \in \mathbf{U}.$$

We denote by $\partial_{\triangleleft} : \mathbf{U} \rightarrow D_{\mathbf{U}}$ the inclusion into the subalgebra $(1 \otimes \mathbf{U})$. The algebra \mathbf{U}' is a locally finite left \mathbf{U} -module, and thus a right $\tilde{\mathbf{A}}$ comodule algebra, via the adjoint action; we have a linear map: $\text{ad}^* : \mathbf{U}' \rightarrow \mathbf{U}' \otimes \mathbf{A}$, $u \mapsto \sum_i u_i \otimes g_i$. By definition, we have $\sum_i g_i(v)u_i = v_{(1)}uS(v_{(2)})$, for all $v \in \mathbf{U}$.

Remark 6.2. For quantum groups defined over formal power series, it is not necessary to distinguish between \mathbf{U} and its locally finite part, as the space of vectors of finite type under the $\text{ad } \mathbf{U}$ -action is dense in the \hbar -adic topology, so for instance the co-adjoint map $\text{ad}^* : \mathbf{U} \rightarrow \mathbf{U} \otimes \tilde{\mathbf{A}}$ is automatically well-defined as a formal power series – this is all we need.

Proposition 6.3. *[VV], Proposition 1.8.2(c), Remark 1.8.4] We have:*

- (1) *If \mathbf{U} has enough finite-dimensional modules (see, e.g. [J], Definition 2.12, Theorem 2.18), then the algebra \mathbf{A} is a faithful representation for $D_{\mathbf{U}}$. (We will make this assumption from now on).*
- (2) *The map $\partial_{\triangleright} : \mathbf{U}' \rightarrow D_{\mathbf{U}}$, given by*

$$\partial_{\triangleright}(u) := \sum_{i,j} ((r_j \otimes 1) \triangleright g_i) \partial_{\triangleleft}(S^{-1}(u_i r'_j)),$$

is a homomorphism of algebras.

- (3) *The algebra $\tilde{\mathbf{A}}$ is equivariant for the resulting $\mathbf{U} \otimes \mathbf{U}'$ action.*
- (4) *The images $\partial_{\triangleleft}(\mathbf{U})$ and $\partial_{\triangleright}(\mathbf{U}')$ commute in $D_{\mathbf{U}}$, so we have a homomorphism $\partial_2 = \partial_{\triangleleft} \otimes \partial_{\triangleright} : \mathbf{U} \otimes \mathbf{U}' \rightarrow D_{\mathbf{U}}$.*
- (5) *∂_2 is a quantum moment map: on generators $V^* \boxtimes V$ of \mathbf{A} , the $\mathbf{U} \otimes \mathbf{U}'$ -action is given by:*

$$\partial_2(x \otimes y)(f \boxtimes v) = xf \boxtimes yv.$$

Proof. The proof of 6.3.1 given in [J] applies as well here. We include proofs of 6.3.2-5 for the reader’s convenience, as our conventions differ slightly from [VV].

For 6.3.2, we claim that $\partial_{\triangleright}(u)f = (1 \otimes u) \triangleright f = f(\bullet u)$, for all $f \in \tilde{\mathbf{A}}$. Here $f(\bullet u)(v) := f(vu)$. This will imply that $\rho_{\tilde{\mathbf{A}}} \circ \partial_{\triangleright}$, and thus $\partial_{\triangleright}$ by claim 6.3.1, is a

homomorphism. Thus we let $v \in \mathbf{U}$ and compute:

$$\begin{aligned}
(\partial_{\triangleright}(u)f)(v) &= \left(\sum_{i,j} (r_j \otimes 1) \triangleright g_i \right) \partial_{\triangleleft}(S^{-1}(u_i r'_j))f(v) \\
&= \sum_{i,j} (g_i(S(r_j) -) f(u_i r'_j -))(v) \\
&= \sum_{i,j,k} g_i(S(r_j) r_k v_{(1)}) f(u_i r'_j r'_k v_{(2)}), \text{ by (18)} \\
&= \sum_k g_i(v_{(1)}) f(u_i v_{(2)}) \\
&= f(v_{(1)} u S(v_{(2)}) v_{(3)}) \\
&= f(vu),
\end{aligned}$$

as claimed. The proof of 6.3.3 is clear, because $\tilde{\mathbf{A}}$ was constructed as an algebra in $\mathcal{C} \boxtimes \mathcal{C}$. To prove 6.3.4, it suffices to consider the $D_{\mathbf{U}}$ -module $\tilde{\mathbf{A}}$, by claim 6.3.1. We have

$$(\partial_{\triangleright}(u_1) \partial_{\triangleleft}(u_2) f)(v) = f(S(u_2) v u_1) = (\partial_{\triangleleft}(u_2) \partial_{\triangleright}(u_1) f)(v).$$

Claim 6.3.5 follows from the proof of claim 6.3.2. \square

Remark 6.4. We will make repeated use of the faithfulness of $\tilde{\mathbf{A}}$ in coming sections, as in the proofs of 6.3.2 and 6.3.4 above. In particular, the proofs of Proposition 6.8 and Theorem 9.1 require us to check certain relations amongst elements in $\text{End}_{\mathbb{C}}(M \otimes U)$, where M is a $D_{\mathbf{U}}$ -module, and U is a finite dimensional vector space. Each relation is of the form $(\rho_M \otimes \text{id})(X)$ for some $X \in D_{\mathbf{U}} \otimes \text{End}_{\mathbb{C}}(U)$, and thus holds for all $D_{\mathbf{U}}$ modules if, and only if, X is already zero. Since \mathbf{A} is faithful, we can verify $X = 0$ by evaluating at $M = \mathbf{A}$.

Remark 6.5. The algebra $\tilde{\mathbf{A}}$ is constructed to be equivariant for a $\mathbf{U} \otimes \mathbf{U}$ -action, while the algebra $F_2(\mathbf{A})$ is equivariant for the diagonal \mathbf{U} -action. In [VV], there is yet another relative of \mathbf{A} , denoted \mathbf{F} , which is equivariant for a $\mathbf{U}^{co-op} \otimes \mathbf{U}$ -action. These algebras are not each isomorphic. However, the smash-product algebras $D_{\mathbf{U}}$ defined from them are isomorphic. See [VV], Proposition 1.4.2 for details.

6.2. Non-degenerate quantum D -modules. Classically, a $D(G)$ module is a module over the algebra $\mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$ via the inclusions of $\mathcal{U}(\mathfrak{g})$ into $D(G)$ by left- and right-invariant differential operators. The quantum analog of these actions are given by the homomorphism $\partial_2 : \mathbf{U}' \otimes \mathbf{U}' \rightarrow D_{\mathbf{U}}$. For $\mathbf{U} = \mathcal{U}(\mathfrak{g})$, we have $\mathbf{U}' = \mathbf{U}$, and this recovers the commuting actions entirely; for more general quasi-triangular Hopf algebras \mathbf{U} (including those coming from quantum groups), it can happen that $\mathbf{U}' \neq \mathbf{U}$.

We thus introduce the following definitions. We denote by $\tilde{\mathcal{C}}$ and $\tilde{\mathcal{D}}$ the categories of \mathbf{U} -modules and \mathbf{U}' -modules, respectively. We denote by $\text{Res} : \tilde{\mathcal{C}} \rightarrow \tilde{\mathcal{D}}$ the functor of restriction.

Definition 6.6. A non-degenerate $D_{\mathbf{U}}$ -module M is an object of $\tilde{\mathcal{C}} \boxtimes \tilde{\mathcal{C}}$, together with the structure of a $D_{\mathbf{U}}$ -module on $(\text{id} \boxtimes \text{Res})(M)$, such that the two actions of $\mathbf{U} \otimes \mathbf{U}'$ coincide.

Remark 6.7. In other words, we ask for an extension of the action of $\partial_{\triangleright}(\mathbf{U}')$ to an action of \mathbf{U} . For a general quasi-triangular Hopf algebra, it is not completely clear how many $D_{\mathbf{U}}$ modules admit non-degenerate structure. However, see Section 7.3.

6.3. Construction of the representations. Let M be a non-degenerate $D_{\mathbf{U}}$ -module. Let f, g be two characters of \mathbf{A} , and let $J := J_V$ be the numerical solution to the right-handed reflection equation for f , and $J' := J'_V$ be the numerical solution to the left-handed reflection equation for g . Let $\chi : \mathbf{B}_f \rightarrow \mathbb{C}$ be a character, and let 1_χ denote the associated one dimensional representation. We regard any non-degenerate $D_{\mathbf{U}}$ -module M as a $\mathbf{U} \otimes \mathbf{U}$ -module via the homomorphism ∂_2 of Section 6.1, which we extend to $\mathbf{U} \otimes \mathbf{U}$. We then define (reusing the previous notation):

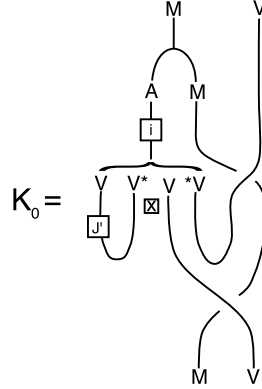
$$F_{n,V}^{f,\chi,g} := \text{Hom}_{\mathbf{B}_f}(1_\chi, M \otimes_2 (1 \boxtimes V_1) \otimes_2 \cdots \otimes_2 (1 \boxtimes V_n)).$$

In other words, we regard each V_i as an object in $\mathcal{C} \boxtimes \mathcal{D}$, i.e. a $\mathbf{U} \otimes \mathbf{U}'$ -module with trivial action in the first components. Here \mathbf{B}_f acts on the tensor product via the restriction of the homomorphism $\partial_{\triangleright} : \mathbf{B}_f \rightarrow D_{\mathbf{U}}$. We let $\hat{\mathcal{B}}_n$ act as before, acting always on the second tensor component (which means it acts by *left translation*, which are *right-invariant* quantum vector fields on M).

We define the following operator

$$(19) \quad K_0 := \mu_M \circ \sigma_{1 \boxtimes V, M} \circ ((J' \otimes 1) \circ \text{coev}_V \boxtimes (\text{id} \otimes \text{coev}_{*V})) \circ \sigma_{1 \boxtimes V_1, M}^{-1},$$

depicted in the following figure:



K_0 is thus constructed from \mathbf{B}_f -linear (\mathbf{U} -linear, even) morphisms on the second \boxtimes -component, and so it automatically preserves spaces of \mathbf{B}_f -invariants.

Proposition 6.8. *We have following identity:*

$$T_1 K_0 T_1 K_0 = K_0 T_1 K_0 T_1, \text{ and } K_0 T_i = T_i K_0 \text{ for } i \geq 2.$$

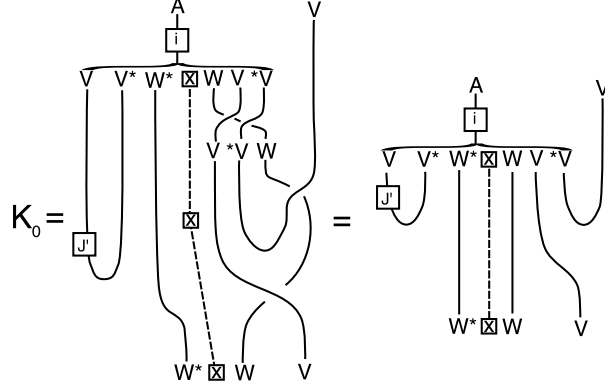
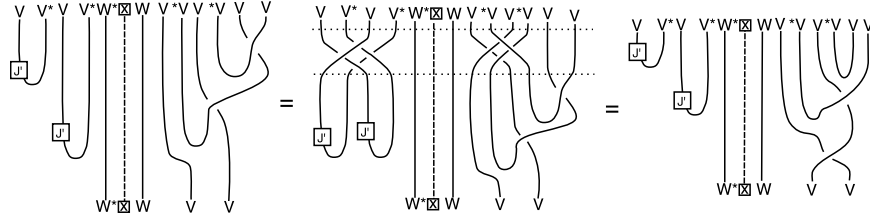
Proof. The second set of relations is clear because in this case T_i and K_0 act on distinct tensor factors. To show the first relation, we will compute it explicitly in the case $M = \mathbf{A}$, as in Remark 6.4. For this, we can explicitly compute the multiplication $\mu_M = \mu_{\mathbf{A}}$ on the generating subspaces $W^* \boxtimes W$ of \mathbf{A} , where K_0 takes the simpler form of Figure 5.

In Figure 6, we prove the relation $T_1 K_0 T_1 K_0 = K_0 T_1 K_0 T_1$. □

It remains to show relation (3) in Definition 2.3.

Lemma 6.9. *On the space of χ -invariants, we have the identity*

$$T_0^{-1} = \sigma_{V,M} \tilde{J}_{V_1} \sigma_{V_1,M}^{-1}, \text{ where } \tilde{J} = \sum E_i^l \rho_V(S(l_{ij}^+ \chi(c_{jk}) S(l_{kl}^-))).$$

FIGURE 5. K_0 acting on the generating subspace $W^* \boxtimes W$ of \mathbf{A} .FIGURE 6. Proof of $T_1 K_0 T_1 K_0 = K_0 T_1 K_0 T_1$. The first equality applies the relations in equation (4) between the dotted lines, noting that the two tangles appearing there are adjoint-inverse to one another. The second equality applies QYBE and the left-handed reflection equation for J' .

Proof. We compute:

$$\begin{aligned}
 T_0^{-1} &= \sigma_{V,M} \sigma_{M,V} T_{(1 \dots n)} T_n T_{(n \dots 1)} \\
 &= \sigma_{V,M} \sigma_{M,V} T_{(1 \dots n)} T_n T_{(n \dots 1)} \sigma_{V,M} \sigma_{V,M}^{-1} \\
 &= \sigma_{V,M} \left(\sum_{i,l} (E_i^l)_{V_1} \otimes (c_{il})_{M \otimes V_2 \otimes \dots \otimes V_n} \right) \sigma_{V,M}^{-1} \\
 &= \sigma_{V,M} \left(\sum E_i^l \rho_V(S(l_{ij}^+ \chi(c_{jk}) S(l_{kl}^-))) \right)_{V_1} \sigma_{V,M}^{-1},
 \end{aligned}$$

as desired. In the final equality, we have applied the identity

$$(1 \otimes x) = (S(x_{(1)}) \otimes 1)(x_{(2)} \otimes x_{(3)}) = (S(x_{(1)}) \chi(x_{(2)}) \otimes 1)$$

to $x = c_{il}$, using the left coideal property for \mathbf{B}_f . \square

The final relation (3) of Definition 2.3 is computed in Figure 7. We have proven the following:

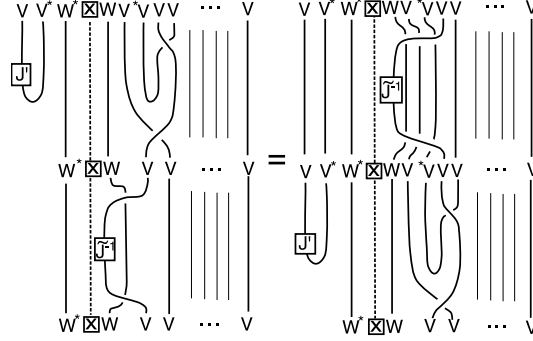


FIGURE 7. Proof of relation $T_1^{-1}K_0T_1T_0 = T_0T_1^{-1}K_0T_1$. We have applied Lemma 6.9 to simplify the appearance of T_0 in both sides of the equality. The moves from the left hand side to the right hand side are only QYBE.

Theorem 6.10. *The operators T_0, \dots, T_n and K_0 define a representation of $\tilde{\mathcal{B}}_n$ on $F_{n,V}^{f,X,g}(M)$. We have a functor:*

$$F_{n,V}^{f,X,g} : \begin{array}{c} \text{non-degenerate} \\ D_{\mathbf{U}}\text{-modules} \end{array} \longrightarrow \tilde{\mathcal{B}}_n\text{-mod.}$$

7. QUANTUM GROUPS AND QUANTUM SYMMETRIC PAIRS

7.1. The Drinfeld-Jimbo quantum group $\mathcal{U}_q(\mathfrak{gl}_N)$ and its representations.

Let $\mathfrak{g} = \mathfrak{gl}(N, \mathbb{C})$ be the Lie algebra of general linear algebraic group $G = \mathrm{GL}(N, \mathbb{C})$. Let $\mathcal{E}_N = \mathbb{R}^N$, with standard basis ε_i and inner product $(\varepsilon_i, \varepsilon_j) = \delta_{ij}$. Let $\Pi_+^{A_{N-1}} = \{\alpha_i = \varepsilon_i - \varepsilon_{i+1} \mid i = 1, \dots, N-1\}$ be the set of simple roots of \mathfrak{g} and $\Pi^{A_{N-1}}$ be the set of roots. Let Λ (resp. Λ^+) be the set of integral (dominant integral) weights for \mathfrak{gl}_N :

$$\Lambda = \{m_1\varepsilon_1 + \dots + m_N\varepsilon_N \mid m_i \in \mathbb{Z}\}.$$

$$\Lambda^+ = \{m_1\varepsilon_1 + \dots + m_N\varepsilon_N \mid m_i \in \mathbb{Z}, m_1 \geq \dots \geq m_N\}.$$

We let $\omega_N := \varepsilon_1 + \dots + \varepsilon_N$ denote the fundamental weight corresponding to the determinant representation.

Let $q \in \mathbb{C}^\times$ be a nonzero complex number and assume q is not a root of unity. Set $E_i := E_{\alpha_i}$ and $F_i := F_{\alpha_i}$ for each simple root. Then the Drinfeld-Jimbo algebra $\mathcal{U}_q(\mathfrak{g})$ is generated by elements E_i, F_i , ($1 \leq i \leq N-1$), and K_j, K_j^{-1} ($1 \leq j \leq N$), with relations:

$$\begin{aligned} K_i K_j - K_j K_i &= 0, & K_i K_i^{-1} &= K_i^{-1} K_i = 1, \\ K_i E_j K_i^{-1} &= q^{\delta_{i,j} - \delta_{i,j+1}} E_j, & K_i F_j K_i^{-1} &= q^{-\delta_{i,j} + \delta_{i,j+1}} F_j, \\ E_i F_j - F_j E_i &= \delta_{i,j} \frac{K_i K_{i+1}^{-1} - K_i^{-1} K_{i+1}}{q - q^{-1}}, \\ E_i E_j - E_j E_i &= 0, & F_i F_j - F_j F_i &= 0, & |i - j| &\leq 2, \\ E_i^2 E_{i\pm 1} - (q + q^{-1}) E_i E_{i\pm 1} E_i + E_{i\pm 1} E_i^2 &= 0, \\ F_i^2 F_{i\pm 1} - (q + q^{-1}) F_i F_{i\pm 1} F_i + F_{i\pm 1} F_i^2 &= 0. \end{aligned}$$

For any $\lambda \in \Lambda$ with $\lambda = \sum_i n_i \varepsilon_i$, we will denote $K^\lambda := K_1^{n_1} \cdots K_N^{n_N}$. The Hopf structure on $\mathcal{U}_q(\mathfrak{g})$ is given by:

$$\begin{aligned} \Delta(K_i^\pm) &= K_i^\pm \otimes K_i^\pm, \quad \Delta(E_i) = E_i \otimes K_i K_{i+1}^{-1} + 1 \otimes E_i, \\ \Delta(F_i) &= F_i \otimes 1 + K_i^{-1} K_{i+1} \otimes F_i, \quad \epsilon(K_i) = 1, \quad \epsilon(E_i) = \epsilon(F_i) = 0, \\ S(K_i) &= K_i^{-1}, \quad S(E_i) = -E_i K_i^{-1} K_{i+1}, \quad S(F_i) = -K_i K_{i+1}^{-1} F_i. \end{aligned}$$

We will consider the block of type I $\mathcal{U}_q(\mathfrak{g})$ -modules, where the generators K_i act on a vector v of weight μ by $q^{\langle \varepsilon_i, \mu \rangle}$. See [KLSch] for details.

7.2. The vector representation of $\mathcal{U}_q(\mathfrak{g})$. Now let e_i be the standard basis for $V = \mathbb{C}^N$. The vector representation ρ_V of $\mathcal{U}_q(\mathfrak{g})$ on $V = \mathbb{C}^N$ is given by:

$$\begin{aligned} \rho_V(K_i) &= q^{-1} E_i^i + \sum_{i \neq j} E_j^j, \quad i = 1, \dots, N, \\ \rho_V(E_i) &= E_{i+1}^i, \quad \rho_V(F_i) = E_i^{i+1}, \quad i = 1, \dots, N-1. \end{aligned}$$

The R -matrix for the vector representation can be expressed explicitly:

$$(20) \quad R := (\rho_V \otimes \rho_V) \circ \mathcal{R} = q \sum_i E_i^i \otimes E_i^i + \sum_{i \neq j} E_i^i \otimes E_j^j + (q - q^{-1}) \sum_{i > j} E_i^j \otimes E_j^i.$$

We define $R_{jl}^{ik}, (R^{-1})_{jl}^{ik} \in \mathbb{C}$, for $i, j, k, l = 1, \dots, N$ by

$$R(e_i \otimes e_j) = \sum_{i,j} R_{ij}^{kl} (e_k \otimes e_l), \quad R^{-1}(e_i \otimes e_j) = \sum_{i,j} (R^{-1})_{ij}^{kl} (e_k \otimes e_l).$$

We can write the coefficients explicitly as follows:

$$(21) \quad R_{ij}^{kl} = \begin{cases} q, & i = j = k = l; \\ 1, & i = k \neq j = l; \\ q - q^{-1}, & i = l < j = k; \\ 0, & \text{otherwise;} \end{cases} \quad (R^{-1})_{ij}^{kl} = \begin{cases} q^{-1}, & i = j = k = l; \\ 1, & i = k \neq j = l; \\ q^{-1} - q, & i = l < j = k; \\ 0, & \text{otherwise.} \end{cases}$$

We will use the notation L^\pm and l_{ij}^\pm for L_V^\pm and $l_{ij}^{V,\pm}$, when V is the vector representation. The elements l_{ij}^\pm satisfy the following relations:

$$(22) \quad L_1^\pm L_2^\pm R = R L_2^\pm L_1^\pm, \quad L_1^- L_2^+ R = R L_2^+ L_1^-,$$

$$(23) \quad l_{ii}^+ l_{ii}^- = l_{ii}^- l_{ii}^+ = 1, \quad i = 1, \dots, N,$$

$$(24) \quad l_{ij}^+ = l_{ji}^- = 0, \quad i > j.$$

Here $L_\pm = (l_{ij}^\pm)$ and $L_1^\pm = L^\pm \otimes \text{id}$, $L_2^\pm = \text{id} \otimes L^\pm$ which are $N^2 \times N^2$ matrices. In fact, we have the following theorem.

Theorem 7.1 (See e.g. [KLSch], Ch. 8). *The Drinfeld-Jimbo algebra $\mathcal{U}_q(\mathfrak{g})$ is generated by the l_{ij}^\pm , $i, j = 1, \dots, n$, with relations (22), (23), and (24). The antipode S , coproduct Δ and counit ϵ are given by*

$$S(L^\pm) = (L^\pm)^{-1}, \quad \Delta(l_{ij}^\pm) = \sum_k l_{ik}^\pm \otimes l_{kj}^\pm, \quad \text{and} \quad \epsilon(l_{ij}^\pm) = \delta_{ij}.$$

By their definition, the elements l_{ij}^\pm act on $V = \mathbb{C}^N$ via the R -matrix; more precisely, we have

$$\rho_V(l_{ij}^+) = \sum_{k,l} R_{lj}^{ki} E_k^l, \quad \rho_V(l_{ij}^-) = \sum_{k,l} (R^{-1})_{jl}^{ik} E_k^l.$$

7.3. Non-degenerate quantum D -modules for $\mathcal{U}_q(\mathfrak{gl}_N)$. In Section 6.2, we have introduced the notion of non-degeneracy for $D_{\mathbf{U}}$ -modules. This condition is necessary for technical reasons; however, in this section we show that the restriction is a mild one in the case $\mathbf{U} = \mathcal{U}_q(\mathfrak{gl}_N)$ (which we assume in this section).

Proposition 7.2. \mathbf{U} is generated as an algebra by \mathbf{U}' and K_1, \dots, K_N .

Proof. Recall that $x \triangleright y := x_{(1)}yS(x_{(2)})$ denotes the adjoint action of \mathbf{U} on itself. We will use the following theorem due to A. Joseph and G. Letzter. Note that we use slightly different conventions for the K_i , and that the statement is adapted to account for the central element $K_{\omega_N} \in \mathcal{U}_q(\mathfrak{gl}_N)$.

Theorem 7.3 (see [JL], Theorem 4.10).

$$\mathbf{U}' = \bigoplus_{\lambda \in -2\Lambda^+ + \mathbb{Z}\omega_N} \mathcal{U}_q(\mathfrak{g}) \triangleright K^\lambda.$$

Now let \mathbf{U}'' be the algebra generated by \mathbf{U}' and K_1, \dots, K_N . It is easy to see that $K_1^{\pm 1}, \dots, K_N^{\pm 1} \in \mathbf{U}''$. For $\lambda \in -2\Lambda^+$ and $i = 1, \dots, n$, we have:

$$\begin{aligned} E_i \triangleright K^\lambda &= E_i K^\lambda K_{i+1} K_i^{-1} - K^\lambda E_i K_i^{-1} K_{i+1} = (1 - q^{(\alpha_i, \lambda)}) E_i K^\lambda K_{i+1} K_i^{-1}, \\ F_i \triangleright K^\lambda &= F_i K^\lambda - K^\lambda F_i = (1 - q^{-(\alpha_i, \lambda)}) F_i K^\lambda. \end{aligned}$$

Thus E_i and $F_i \in \mathbf{U}''$ as well, and so $\mathbf{U}'' = \mathbf{U}$. \square

It follows that \mathbf{U} is obtained from \mathbf{U}' in a two-step process: first one localizes \mathbf{U}' at its denominator set generated by the $K^{-2\alpha_i}$, and then one adjoins a square root K^{α_i} of each $K^{2\alpha_i}$.

7.4. The classical symmetric pair and quantum symmetric pair. Let \mathfrak{g} be a reductive Lie algebra with Cartan decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$. Suppose we have an involution of \mathfrak{g} , denoted by θ . Let $\mathfrak{k} = \mathfrak{g}^\theta$ be the fixed Lie subalgebra in \mathfrak{g} under the involution. Then the pair $(\mathfrak{g}, \mathfrak{k})$ is called a (classical) symmetric pair.

Our primary example of a symmetric pair is constructed as follows. Let $\mathfrak{g} = \mathfrak{gl}(N)$ with $N = p + q$. Let θ be the involutive automorphism of \mathfrak{g} defined by $\theta(u) := JuJ$ where

$$J = \sum_{1 \leq k \leq p} E_k^k - \sum_{p+1 \leq k \leq N} E_k^k.$$

The corresponding Lie subalgebra \mathfrak{k} is $\mathfrak{gl}(p) \times \mathfrak{gl}(q)$ and we get the symmetric pair $(\mathfrak{gl}(N), \mathfrak{gl}(p) \times \mathfrak{gl}(q))$. For our purpose, we would like to consider another symmetric pair $(\mathfrak{g}, \mathfrak{k}')$ as in [DS]. The involution θ' of this symmetric pair is given by $\theta'(u) = J'uJ'$ with

$$(25) \quad J' = \sum_{1 \leq k \leq p} E_k^{N-k+1} + \sum_{1 \leq k \leq p} E_{N-k+1}^k - \sum_{p < k < N-p+1} E_k^k.$$

It is easy to see that \mathfrak{k} and \mathfrak{k}' are conjugate to each other by the matrix g of equation (35).

The theory of quantum symmetric pairs provides an analog of classical symmetric pairs in the setting of quantum groups. It was developed systematically by G. Letzter in a series of papers [L1, L2], with many examples coming from so-called Noumi coideal subalgebras [N, NS, OS].

Let $(\mathfrak{g}, \mathfrak{k})$ denote a classical symmetric pair. A *quantum symmetric pair* associated to $(\mathfrak{g}, \mathfrak{k})$ is a pair $(\mathcal{U}_q(\mathfrak{g}), \mathcal{I})$, where \mathcal{I} is a right coideal subalgebra in $\mathcal{U}_q(\mathfrak{g})$,

such that the quasi-classical limit as $q \rightarrow 1$ recovers $\mathcal{U}(\mathfrak{k})$. The coideal formalism arises because while $\mathcal{U}(\mathfrak{k})$ is a sub-Hopf algebra of $\mathcal{U}(\mathfrak{g})$, the quantization \mathcal{I} of $\mathcal{U}(\mathfrak{k})$ inside $\mathcal{U}_q(\mathfrak{g})$ is no longer a sub-coalgebra, but only a one-sided coideal.

7.5. The one parameter family of coideal subalgebras. The symmetric pair $(\mathfrak{g}, \mathfrak{k}')$ can be quantized via the method of characters $f : F_2(\mathbf{A}) \rightarrow \mathbb{C}$, where $F_2(\mathbf{A})$ is the braided dual of $\mathcal{U}_q(\mathfrak{gl}_N)$. Let $\{a_j^i\}$ be the generators of $F_2(\mathbf{A})$ which are defined in Section 4.2. Characters for the reflection equation algebra associated to $\mathcal{U}_q(\mathfrak{gl}_N)$ were studied by Donin, Kulish and Mudrov [DKM, DM1, DM2], and completely classified in [Mud]. In [KoSt], it was explained that a character f of the reflection equation algebra extends to a character of the braided dual of $\mathcal{U}_q(\mathfrak{gl}_N)$ if, and only if, the matrix $(f(a_j^i))$ is invertible. Following them (see also [N, OS, DS]), we choose⁷ $q^\sigma \in \mathbb{C}$, and define an $N \times N$ complex matrix J^σ :

$$(26) \quad J^\sigma = \sum_{1 \leq k \leq p} (q^\sigma - q^{-\sigma}) E_k^k - \sum_{p < k < N-p+1} q^{-\sigma} E_k^k + \sum_{1 \leq k \leq p} E_k^{N-k+1} + \sum_{1 \leq k \leq p} E_{N-k+1}^k.$$

Note that J^σ satisfies a Hecke relation $J^\sigma \sim q^\sigma$.

Lemma 7.4 (See e.g. [DS], [DNS], [Mud]). *The matrix J^σ is a right-handed numerical solution of the reflection equation*

$$(27) \quad R_{21} J_1^\sigma R_{12} J_2^\sigma = J_2^\sigma R_{21} J_1^\sigma R_{12},$$

where $J_1^\sigma = J^\sigma \otimes \text{id}$ and $J_2^\sigma = \text{id} \otimes J^\sigma$.

Corollary 7.5. *The matrix $(J^\sigma)^{-1}$ is a left-handed numerical solution of the reflection equation.*

Proof. By the lemma, J^σ is a solution of the right handed reflection equation for all $q^\sigma \in \mathbb{C}$. Let us write $R = R(q)$ and $J^\sigma = J^\sigma(q)$ to emphasize the dependence on q . By inspecting the R -matrix for $V \otimes V$, we see that $R(q)^{-1} = R(q^{-1})$. Similarly $J^\sigma(q) = J^{-\sigma}(q^{-1})$. Thus, we compute that the left handed reflection equation for $J^{-\sigma}$ at q is equivalent to the left-handed equation for $(J^\sigma)^{-1}$ at q^{-1} :

$$\begin{aligned} & R_{21}(q) J_1^{-\sigma}(q) R_{12}(q) J_2^{-\sigma}(q) = J_2^{-\sigma}(q) R_{21}(q) J_1^{-\sigma}(q) R_{12}(q) \\ \Leftrightarrow & R_{21}(q^{-1})^{-1} J_1^\sigma(q^{-1}) R_{12}(q^{-1})^{-1} J_2^\sigma(q^{-1}) = J_2^\sigma(q^{-1}) R_{21}(q^{-1})^{-1} J_1^\sigma(q^{-1}) R_{12}(q^{-1})^{-1}, \\ \Leftrightarrow & J_2^\sigma(q^{-1})^{-1} R_{12}(q^{-1}) J_1^\sigma(q^{-1})^{-1} R_{21}(q^{-1}) = R_{12}(q^{-1}) J_1^\sigma(q^{-1})^{-1} R_{21}(q^{-1}) J_2^\sigma(q^{-1})^{-1}, \\ \Leftrightarrow & J_1^\sigma(q^{-1})^{-1} R_{21}(q^{-1}) J_2^\sigma(q^{-1})^{-1} R_{12}(q^{-1}) = R_{21}(q^{-1}) J_2^\sigma(q^{-1})^{-1} R_{12}(q^{-1}) J_1^\sigma(q^{-1})^{-1}. \end{aligned}$$

The first equivalence follows from the preceding paragraph. The second is by inverting both sides of the equation, and the third is by applying the flip τ_{12} . Since the right handed reflection equation is established for $J^\sigma(q)$ at all parameters q and q^σ , it follows that the left hand reflection equation holds for $J^\sigma(q)$ for all q and q^σ as well. \square

Thus we can define characters $f_\sigma : F_2(\mathbf{A}) \rightarrow \mathbb{C}$, $f_\sigma(a_i^j) := J_{ij}^\sigma$, and $g_\rho : F_2(\mathbf{A}) \rightarrow \mathbb{C}$, $g_\rho(\tilde{a}_i^j) = ((J^\rho)^{-1})_{ij}$. Note that the corresponding matrices $J_V := \sum f(a_j^i) E_i^j$ and $J'_V := \sum g(\tilde{a}_j^i) E_i^j$ for the vector representation $V = \mathbb{C}^N$ will be J^σ and $(J^\rho)^{-1}$

⁷In this article q^σ denotes a generic complex number, not directly related to q . We keep the old notation for two reasons: first to emphasize the connection with previous papers [DS, NS, OS], and second, because in the formal setting we will take $\sigma \in \mathbb{C}$, and let $q := e^h$, and $q^\sigma := e^{\sigma h}$, in order to compute the trigonometric degeneration. We let $q^{-\sigma} := \frac{1}{q^\sigma}$.

themselves, since J^σ and $(J^\rho)^{-1}$ are symmetric. Following section 4.4, we have coideal subalgebras $\mathbf{B}_\sigma := \mathbf{B}_{f_\sigma}$ and $\mathbf{B}'_\rho := \mathbf{B}'_{g_\rho}$ associated to any $V \in \mathcal{C}$.⁸

In Letzter's framework [L1, L2], it is important that the coideal subalgebras \mathbf{B}_σ are all isomorphic as abstract algebras (similarly for the \mathbf{B}'_ρ). This property was also used in [OS] in the case $p = q$, where the authors constructed a single comodule algebra and a family of embeddings into the quantum group. In our case, the isomorphisms between the \mathbf{B}_σ take an especially simple form in the following proposition:

Proposition 7.6. *Let $\mathbf{q}, \mathbf{q}^{\sigma_1}, \mathbf{q}^{\sigma_2} \in \mathbb{C}$ be generic, and let $\phi : \mathbf{B}_{\sigma_1} \rightarrow \mathbf{B}_{\sigma_2}$ be defined on generators by $\phi(c_{il}^{(1)}) = c_{il}^{(2)}$, where $c_{il}^{(k)}$ are the generators (15) for \mathbf{B}_{σ_k} . Then ϕ is an isomorphism of algebras.*

Proof. Using that L^+ (resp. L^-) is upper (resp. lower) triangular, that $S(l_{ii}^-) = l_{ii}^+$, and that J^σ is skew-upper triangular and symmetric, we can see by inspection that the matrix of generators (c_{il}) has the form:

$$c_{il} = \begin{pmatrix} * & * & X \\ * & * & 0 \\ Y & 0 & 0 \end{pmatrix}_{il},$$

where the blocks are of size $(p, q-p, p) \times (p, q-p, p)$ (the same as in J^σ). Here, the $*$'s are some nonzero expressions, X and Y are skew upper triangular, and we have $X_{i,p-i} = Y_{p-i,i}$. This means that each \mathcal{I}_σ is really generated by the q^2 entries in the $*$ 'ed regions, plus the p^2 entries in X and Y , counting the diagonal only once. This gives a system of $p^2 + q^2$ generators, which are subject to (at least) the relations of the reflection equation algebra:

$$(28) \quad R_{21}c_1R_{12}c_2 = c_2R_{21}c_1R_{12}.$$

It follows that the algebras \mathbf{B}_σ are spanned by ordered monomials in the c_{il} , though *a priori* we may expect more relations.

It turns out that there are no other relations, which we can see as follows. It is shown in Section 10.4 that the quasi-classical limits of the elements c_{il} are the generators of the subalgebra $\mathcal{U}(\mathfrak{k}) = \mathcal{U}(\mathfrak{k}') \subset \mathcal{U}(\mathfrak{gl}_N)$, which itself affords a PBW basis of ordered monomials in its generators. It now follows from the fact that $\mathcal{U}_{\mathbf{q}}(\mathfrak{g})$ is a flat deformation of $\mathcal{U}(\mathfrak{g})$, for \mathbf{q} not a root of unity, that the relations (28) provide all the relations on \mathbf{B}_σ . In particular, the relations don't depend at all on \mathbf{q}^σ , so the map ϕ is an isomorphism. \square

Obviously the map $\chi_\sigma : c_{il} \mapsto J_{il}^\sigma$ is a character of \mathbf{B}_σ (χ_σ is the restriction of ϵ). In fact, we see by the previous proposition that each \mathbf{B}_σ has a two parameter family of characters:

$$(29) \quad \chi_\tau^\eta(l_{ij}^+ J_{jk}^\sigma S(l_{kl}^-)) := \mathbf{q}^\eta J_{il}^\tau.$$

Likewise, each \mathbf{B}'_ρ has a two parameter family of characters:

$$(30) \quad \lambda_\nu^\omega(S(l_{ij}^-)(J^\rho)^{-1}_{jk}l_{kl}^+) := \mathbf{q}^\omega(J^\nu)^{-1}_{il}.$$

In the next two sections, we will use these to construct twisted invariants and twisted quantum D -modules.

⁸It is also possible to scale the matrices J^σ by an arbitrary nonzero complex number. Of course, doing so will yield the same algebra.

7.6. \mathbf{q} -Harish Chandra modules for $(\mathcal{U}_{\mathbf{q}}(\mathfrak{gl}_N), \mathbf{B}_{\sigma})$. In the theory of real and p -adic groups, an important role is played by the so-called Harish-Chandra modules associated to a symmetric pair (G, K) . The following definition captures the relevant algebraic properties in the \mathbf{q} -deformed setting, and was proposed in [L3], Definition 3.1.

Definition 7.7. The category of \mathbf{q} -Harish-Chandra modules for $(\mathcal{U}_{\mathbf{q}}(\mathfrak{gl}_N), \mathbf{B}_{\sigma})$ is the full abelian subcategory of $\mathcal{U}_{\mathbf{q}}(\mathfrak{gl}_N)$ -modules M such that \mathbf{B}_{σ} acts semi-simply on M .

Definition 7.8. The category of \mathbf{q} -Harish-Chandra D -modules for $(\mathcal{U}_{\mathbf{q}}(\mathfrak{gl}_N), \mathbf{B}_{\sigma}, \mathbf{B}'_{\rho})$ is the full abelian subcategory of non-degenerate $D_{\mathcal{U}_{\mathbf{q}}(\mathfrak{gl}_N)}$ -modules M such that $\partial_2(\mathbf{B}'_{\rho} \otimes \mathbf{B}_{\sigma})$ acts semi-simply on M .

In either case, we have the “Harish-Chandra part” functor which sends a module to sum of all its \mathbf{q} -Harish Chandra submodules; the result is only a $\mathbf{U}'\mathbf{B}_{\sigma}$ -module (see the discussion in [L3] following Definition 3.1). In the case of non-degenerate $D_{\mathbf{U}}$ -modules, the Harish-Chandra part is only a $\mathbf{U}'\mathbf{B}'_{\rho} \otimes \mathbf{U}'\mathbf{B}_{\sigma}$ module, which is preserved by the \mathbf{A} action. This is enough for our purposes.

8. REPRESENTATIONS OF THE AFFINE HECKE ALGEBRAS OF TYPE $C^{\vee}C_n$.

Let $V = \mathbb{C}^N$ be the vector representation for $\mathcal{U}_{\mathbf{q}}(\mathfrak{g}) = \mathcal{U}_{\mathbf{q}}(\mathfrak{gl}_N)$. Let χ_{τ}^{η} be the character of \mathbf{B}_{σ} defined in (29), and let $\mathbb{1}_{\tau}^{\eta}$ denote the associated one-dimensional representation. For any \mathbf{B}_{σ} -module W , we denote by $W^{l.f.}$ the locally finite part of W , i.e. the sum of all finite dimensional \mathbf{B}_{σ} -submodules of W . For any $\mathcal{U}_{\mathbf{q}}(\mathfrak{g})$ -module M , define a vector space

$$F_n^{\sigma, \eta, \tau}(M) = (M \otimes V^{\otimes n})^{\mathbf{B}_{\sigma}, \chi_{\tau}^{\eta}} := \text{Hom}_{\mathbf{B}_{\sigma}}(\mathbb{1}_{\tau}^{\eta}, M \otimes V^{\otimes n}).$$

Above, the \mathbf{B}_{σ} action on the tensor product is as in Section 6. The main result of this section is the following theorem.

Theorem 8.1. $F_n^{\sigma, \eta, \tau}$ defines an functor from the category of $\mathcal{U}_{\mathbf{q}}(\mathfrak{g})$ -modules to the category of representations of the affine Hecke algebra $\mathcal{H}_n(t, t_0, t_n)$ with parameters:

$$t = \mathbf{q}, \quad t_n = \mathbf{q}^{\sigma}, \quad t_0 = \mathbf{q}^{(p-q-\tau)}.$$

Moreover $F_n^{\sigma, \eta, \tau}$ factors through the Harish-Chandra part functor, and is exact on the category of \mathbf{q} -Harish-Chandra modules.

The construction is a specialization of Section 5, except that we rescale the operators to have eigenvalues of the form $\lambda, -\lambda^{-1}$. It is clear that the relations we checked in Section 5 are unchanged by rescaling; thus, the only new proofs in this section will be checking the Hecke relations.

For $i = 1, \dots, n-1$, we let $T_i = \sigma_{V_i, V_{i+1}}$, and we let $T_n = J_{V_n}^{\sigma}$. We let $T_0 = \alpha P_1^{-1}(\sigma_{V_1, M} \circ \sigma_{M, V_1})^{-1}$, where $\alpha = \mathbf{q}^{-N+\eta}$. It follows immediately that $T_i \sim \mathbf{q}$, and $T_n \sim \mathbf{q}^{\sigma}$.

Proposition 8.2. $T_0 \sim \mathbf{q}^{p-q-\tau}$.

Proof. By Lemma 6.9, on the space of $(\mathcal{I}_\sigma, \chi_\tau)$ -invariants, T_0^{-1} has the same minimal polynomial as $\alpha^{-1}\tilde{J} = \mathbf{q}^{N-\eta} \sum E_i^l \rho(S(l_{ij}^+ \chi_\tau^\eta(c_{jk}) S(l_{kl}^-)))$. Applying the definition of χ_τ^η , we have:

$$\begin{aligned} \alpha^{-1}\tilde{J} &= \mathbf{q}^N \sum E_i^l \rho(S(l_{ij}^+ J_{jk}^\tau S(l_{kl}^-))) \\ &= \mathbf{q}^N \sum E_i^l \rho(S^2(l_{kl}^-) J_{jk}^\tau S(l_{ij}^+)) \\ &= \mathbf{q}^N \sum E_i^l \rho(ul_{kl}^- u^{-1} J_{jk}^\tau S(l_{ij}^+)), \end{aligned}$$

where u is the Drinfeld element such that $S^2(x) = u x u^{-1}$ for all $x \in \mathbf{U}$. For the vector representation we have the well-known formula⁹:

$$\rho_V(u) = \sum_{i=1}^N \mathbf{q}^{2i-2} E_i^i.$$

By equations (20) and (21) and direct computation, we have

$$\begin{aligned} \alpha^{-1}\tilde{J} &= \sum_{i=1}^p (\mathbf{q}^{q-p+\tau} - \mathbf{q}^{p-q-\tau}) E_i^i - \sum_{i=p+1}^{N-p} \mathbf{q}^{p-q-\tau} E_i^i \\ &\quad + \sum_{i=1}^p \mathbf{q}^{-N+2i-1} E_i^{N+1-i} + \sum_{i=1}^p \mathbf{q}^{N-2i+1} E_{N+1-i}^i, \end{aligned}$$

which is semisimple, with two eigenvalues: $\lambda_1 = \mathbf{q}^{q-p+\tau}$ and $\lambda_2 = -\mathbf{q}^{p-q-\tau}$.

The second part of the theorem follows easily because tensoring is an exact functor, as is $\text{Hom}(\mathbb{1}, \bullet)$, when restricted to the category of \mathbf{q} -Harish-Chandra-modules. \square

9. REPRESENTATIONS OF THE DOUBLE AFFINE HECKE ALGEBRAS OF TYPE $C^\vee C_n$

Let $V = \mathbb{C}^N$ denote the vector representation for $\mathbf{U} = \mathcal{U}_{\mathbf{q}}(\mathfrak{gl}_N)$. Let χ_τ^η and λ_ν^ω be the characters of \mathbf{B}_σ and \mathbf{B}'_ρ , respectively, defined in equations (29) and (30). We denote the corresponding one dimensional representations $\mathbb{1}_\tau^\eta := \mathbb{1}_{\chi_\tau^\eta}$ and $\mathbb{1}_\nu^\omega := \mathbb{1}_{\lambda_\nu^\omega}$. In this section we prove that a certain rescaling of the action defined in Section 6 induces an action of the double affine Hecke algebra of type $C^\vee C_n$. Let M be a non-degenerate $D_{\mathbf{U}}$ -module, and let

$$F_{n,\rho,\omega,\nu}^{\sigma,\eta,\tau}(M) := \text{Hom}_{\mathbf{B}'_\rho \otimes \mathbf{B}_\sigma}(\mathbb{1}_\nu^\omega \boxtimes \mathbb{1}_\tau^\eta, M \otimes_2 (\mathbb{1} \boxtimes V_1) \otimes_2 \cdots \otimes_2 (\mathbb{1} \boxtimes V_n)).$$

Theorem 9.1. $F_{n,\rho,\omega,\nu}^{\sigma,\eta,\tau}$ defines a functor from the category of non-degenerate $D_{\mathbf{U}}$ -modules to the category of representations of the double affine Hecke algebra $\mathcal{H}_n(v, t, t_0, t_n, u_0, u_n)$ with parameters:

$$\begin{aligned} t &= \mathbf{q}, & t_n &= \mathbf{q}^\sigma, & t_0 &= \mathbf{q}^{(p-q-\tau)}, \\ u_0 &= \mathbf{q}^\nu, & u_n &= \mathbf{q}^{-\rho}, & v &= \mathbf{q}^{\eta-N-\omega}. \end{aligned}$$

Moreover $F_{n,\rho,\omega,\nu}^{\sigma,\eta,\tau}$ factors through the \mathbf{q} -Harish-Chandra part of M , and is an exact functor on the category \mathbf{q} -Harish-Chandra $D_{\mathbf{U}}$ -modules.

⁹up to an immaterial scalar, depending on the normalization of u .

We let T_0, \dots, T_n act as in Section 8, and we let K_0 act as in Section 6. We have only to prove the Hecke relations asserted in the theorem. By remark 6.4, we may consider the faithful representation $M = \mathbf{A}$. As in the proof of Proposition 6.8, K_0 takes the explicit form of Figure 5.

Proposition 9.2. *We have the relation $K_0 \sim \mathbf{q}^{-\rho}$.*

Proof. Let $V \in \mathbf{U}\text{-mod}$. In Figure 8, it is proven that the assignment

$$K : \text{End}_{\mathbb{C}}(V) \rightarrow \text{End}_{\mathbb{C}}(A \otimes V)$$

$$X \mapsto (X \otimes \text{id} \otimes \text{id}) \circ (\text{coev}_V \otimes \text{id}) \boxtimes (\text{id} \otimes \text{id} \otimes \text{coev}_{*V})$$

is an algebra homomorphism. It follows that $K_0 = K((J^\rho)^{-1})$ satisfies the same quadratic relation, $K_0 \sim \mathbf{q}^{-\rho}$, as $(J^\rho)^{-1}$. \square

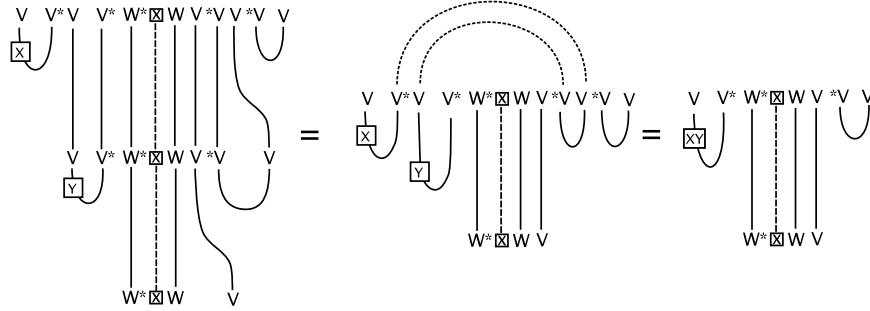


FIGURE 8. Proof of $K(X)K(Y) = K(XY)$. The left hand side is the composition $K(X)K(Y)$. The first equality is straightforward. The second equality applies relations (4) to coev_{*V} as indicated by the dotted lines.

Proposition 9.3. *We have the relation $(vK_0P_1T_0)^{-1} \sim \mathbf{q}^\nu$, where $v = \alpha\mathbf{q}^{-\omega}$.*

Proof. By definition, we have $vK_0P_1T_0 = \mathbf{q}^{-\omega}K_0\sigma_{M,V}^{-1}\sigma_{V,M}^{-1}$. We have the following

Lemma 9.4. *We have the identity:*

$$K_0\sigma_{M,V}^{-1}\sigma_{V,M}^{-1} = \xi \boxtimes (\sigma_{V \otimes^* V, W}^{-1} \circ (\text{id}_V \otimes \text{coev}_{*V})),$$

where $\xi = (\sigma_{V, W^*} \otimes \text{id}) \circ ((J^\rho)^{-1} \otimes \text{id} \otimes \text{id}) \circ (\sigma_{W^*, V} \otimes \text{id}) \circ (\text{id} \otimes \text{coev}_V)$.

Proof. The proof is given in Figure 9. \square

Now, we can express ξ in terms of the c'_{il} :

$$\begin{aligned} \xi : f &\mapsto \sum S(l_{ij}^-)(J^\rho)_{jk}^{-1} l_{kl}^+ f \otimes E_i^l e_m \otimes e^m \\ &= \sum c'_{il} f \otimes E_i^l e_m \otimes e^m, \end{aligned}$$

where $\{e^i\}$ denotes the dual basis to $\{e_i\}$. Thus, on the space of $(B'_\rho, \lambda_\nu^\omega)$ invariants, we have

$$\xi : \sum f_j \boxtimes w_j \otimes v_{j,1} \otimes \dots \otimes v_{j,n} \mapsto \mathbf{q}^\omega \sum f_j \otimes (J^\nu)^{-1} e_m \otimes e^m \boxtimes w_j \otimes v_{j,1} \otimes \dots \otimes v_{j,n}.$$

Thus, we have that

$$\mathbf{q}^{-\omega} K_0 \sigma_{M,V}^{-1} \sigma_{V,M}^{-1} = ((\text{id} \otimes (J^\nu)^{-1} \otimes \text{id}) \boxtimes \text{id}) \circ (\text{id} \otimes \text{coev}_V \boxtimes \sigma_{V \otimes^* V, W}^{-1} \circ \text{coev}_{*V}).$$

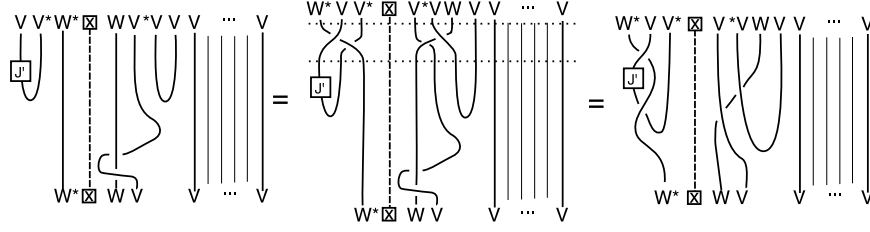


FIGURE 9. Proof of Lemma 9.4. The first equality applies relations of equation (4) between the dotted lines. The second equality uses only QYBE. We have abbreviated $J' := (J^\rho)^{-1}$.

Now arguing as in Proposition 9.2, we see that $vK_0P_1T_0$ has the same minimal polynomial as $(J^\nu)^{-1}$, and we are done.

The second part of the theorem follows as in the proof of Theorem 8.1. \square

Remark 9.5. *A priori*, for each n, N, p , $F_n^{\sigma, \eta, \tau}$ depends upon the four continuous parameters $q, q^\sigma, q^\eta, q^\tau$. However, it is clear from the definition that $F_n^{\sigma, \eta, \tau}$ is the precomposition of $F_n^{\sigma, 0, \tau}$ by the automorphism of \mathcal{C} given by $M \mapsto {}^*1^\eta \otimes M$, corresponding to the fractional tensor power of the determinant character.

A priori, for each n, N, p , $F_{n, \rho, \omega, \nu}^{\sigma, \eta, \tau}$ depends upon the seven continuous parameters, $q, q^\sigma, q^\eta, q^\tau, q^\rho, q^\omega, q^\nu$. However, as above, we can express $F_{\rho, \omega + \xi, \nu}^{n, \sigma, \eta + \xi, \tau}$ as the precomposition of $F_{n, \rho, \omega, \nu}^{\sigma, \eta, \tau}$ by twisting the D_U module M with a fractional tensor power of the determinant local system. On the other hand, $F_{n, \rho, \omega, \nu}^{\sigma, \eta, \tau}(M)$ will be zero unless $\lambda_\nu^\omega(\det_q) = \chi_\tau^\eta(\det_q)q^{-n/N}$. This is because the element \det_q is central and thus its image in D_U under both the left and right actions coincide, so that the values of the characters can only differ by the contribution of the factor $(1 \boxtimes V)^{\otimes n}$. Thus we really have five continuous parameters.

10. THE RELATION TO THE TRIGONOMETRIC dAHA AND dDAHA

In this section we recall the construction in [EFM], and show that it may be recovered as the trigonometric degeneration of our construction. Furthermore, we reprove the main results from that paper, quoted below as Theorems 10.1 and 10.2. Beyond giving a new proof of a known result, this serves two purposes: it provides us an explicit check of our computations in the preceding section, and it also illustrates the process of trigonometric degeneration, whereby very complicated Lie-theoretic formulas appear as the first derivative in \hbar of considerably more natural formulas in quantum groups and braided tensor categories.

10.1. The dAHA of type BC_n . Let $\mathcal{W}_n = \mathcal{S}_n \ltimes (\mathbb{Z}_2)^n$ be the Weyl group of type BC_n . We denote by s_{ij} the reflection in this group corresponding to the root $\varepsilon_i - \varepsilon_j$, and by γ_i the reflection corresponding to ε_i . We abbreviate $s_i := s_{i, i+1}$. The type BC_n dAHA $\mathcal{H}_n^{\deg}(\kappa_1, \kappa_2)$ is generated by y_1, \dots, y_n and $\mathbb{C}[\mathcal{W}_n]$, with cross relations:

$$\begin{aligned} s_i y_i - y_{i+1} s_i &= \kappa_1; & [s_i, y_j] &= 0, & \forall j \neq i, i+1; \\ \gamma_n y_n + y_n \gamma_n &= \kappa_2; & [\gamma_n, y_j] &= 0, & \forall j \neq n; & [y_i, y_j] = 0. \end{aligned}$$

For any $c \neq 0$, we have an isomorphism $\mathcal{H}_n^{\deg}(\kappa_1, \kappa_2) \cong \mathcal{H}_n^{\deg}(c\kappa_1, c\kappa_2)$.

Let us recall the construction of the functor $F_{n,p,\mu}$ in [EFM]. Let \mathbb{C}^N be the vector representation of $\mathfrak{g} = \mathfrak{gl}_N$. Let M be a \mathfrak{g} -module. Define

$$F_{n,p,\mu}(M) = (M \otimes (\mathbb{C}^N)^{\otimes n})^{\mathfrak{k}_0, \mu},$$

where \mathfrak{k}_0 is the subalgebra in $\mathfrak{k} = \mathfrak{gl}_p \times \mathfrak{gl}_q$ consisting of trace zero elements and, for $\mu \in \mathbb{C}$, (\mathfrak{k}_0, μ) -invariants means for all $x \in \mathfrak{k}_0$, $xv = \mu\chi(x)v$. Here χ is a character of \mathfrak{k} defined in [EFM]:

$$(31) \quad \chi\left(\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}\right) = q \operatorname{tr} A_1 - p \operatorname{tr} A_2.$$

The Weyl group \mathcal{W}_n acts on $F_{n,p,\mu}(M)$ in the following way: the element s_{ij} acts by exchanging the i -th and j -th factors, and γ_i acts by multiplying the i -th factor by $J = \begin{pmatrix} I_p & \\ & -I_q \end{pmatrix}$.

Define elements $y_k \in \operatorname{End}_{\mathbb{C}}(F_{n,p,\mu}(M))$ as follows:

$$(32) \quad y_i = - \sum_{s|t} (E_s^t \otimes E_t^s)_{0i} + \frac{p-q-\mu N}{2} \gamma_i + \frac{1}{2} \sum_{k>i} s_{ik} - \frac{1}{2} \sum_{k<i} s_{ik} + \frac{1}{2} \sum_{i \neq k} s_{ik} \gamma_i \gamma_k,$$

where $\sum_{s|t} = \sum_{s=1}^p \sum_{t=p+1}^n + \sum_{t=1}^p \sum_{s=p+1}^n$, the first component acts on M and the second component acts on the k -th factor of the tensor product.

Theorem 10.1 ([EFM]). *The above action of \mathcal{W}_n and the elements y_i define a representation of the degenerate affine Hecke algebra $\mathcal{H}_n^{\deg}(\kappa_1, \kappa_2)$ on the space $F_{n,p,\mu}(M)$, with*

$$\kappa_1 = 1, \quad \kappa_2 = p - q - \mu N.$$

10.2. The dDAHA of type BC_n . The type BC_n dDAHA $\mathcal{H}^{\deg}(t, k_1, k_2, k_3)$ is generated by two commutative families $\{x_i, i = 1, \dots, n\}$, $\{y_i, i = 1, \dots, n\}$ and $\mathbb{C}[\mathcal{W}_n]$, with relations:

- i) $s_i x_i - x_{i+1} s_i = 0$, $[s_i, x_j] = 0$, $(j \neq i, i+1)$;
- ii) $s_i y_i - y_{i+1} s_i = k_1$, $[s_i, y_j] = 0$, $(j \neq i, i+1)$;
- iii) $\gamma_n y_n + y_n \gamma_n = k_2 + k_3$, $\gamma_n x_n = x_n^{-1} \gamma_n$,
 $[\gamma_n, y_j] = [\gamma_n, x_j] = 0$, $(j \neq n)$;
- vi) $[y_j, x_i] = k_1 x_i s_{ij} - k_1 x_i s_{ij} \gamma_i \gamma_j$,
 $[y_i, x_j] = k_1 x_i s_{ij} - k_1 x_j s_{ij} \gamma_i \gamma_j$, $(i < j)$;
- v)

$$\begin{aligned} [y_i, x_i] &= t x_i - k_1 x_i \sum_{k>i} s_{ik} - k_1 \sum_{k<i} s_{ik} x_i - k_1 x_i \sum_{k \neq i} s_{ik} \gamma_i \gamma_k \\ &\quad - (k_2 + k_3) x_i \gamma_i - k_2 \gamma_i. \end{aligned}$$

In particular, we see that the subalgebra in the dDAHA generated by \mathcal{W}_n and the y_i is $\mathcal{H}_n^{\deg}(\kappa_1, \kappa_2)$, where $\kappa_1 = k_1$ and $\kappa_2 = k_2 + k_3$.

Let $\lambda \in \mathbb{C}$. For $x \in \mathfrak{g}$, let L_x denote the vector field on G generated by the left action of x . Let $D^\lambda(\operatorname{GL}(N)/(\operatorname{GL}(p) \times \operatorname{GL}(q)))$ be the sheaf of differential operators on $\operatorname{GL}(N)/(\operatorname{GL}(p) \times \operatorname{GL}(q))$, twisted by the character $\lambda\chi$.

Let M be a $D^\lambda(\operatorname{GL}(N)/(\operatorname{GL}(p) \times \operatorname{GL}(q)))$ -module. Then M is naturally a \mathfrak{g} -module, via the vector fields L_x . Define

$$F_{n,p,\mu}^\lambda(M) = (M \otimes V^{\otimes n})^{\mathfrak{k}_0, \mu}.$$

Then $F_{n,p,\mu}^\lambda(M)$ is a \mathcal{H}_n^{\deg} -module as in the Theorem 10.1.

For $i = 1, \dots, n$, define the following linear operators on the space $F_{n,p,\mu}^\lambda(M)$:

$$x_i = \sum_{s,t} (AJA^{-1}J)_{st} \otimes (E_s^t)_i,$$

where $(AJA^{-1}J)_{st}$ is the function of $A \in \mathrm{GL}(N)/\mathrm{GL}(p) \times \mathrm{GL}(q)$ which takes the st -th element of $AJA^{-1}J$ and the second component acts on the i -th factor in $V^{\otimes n}$.

Theorem 10.2 ([EFM]). *The above action of \mathcal{W}_n and the elements x_i, y_i define a representation of the dDAHA $\mathcal{H}^{\mathrm{deg}}(t, k_1, k_2, k_3)$ on the space $F_{n,p,\mu}^\lambda(M)$, with*

$$(33) \quad t = \frac{2n}{N} + (\lambda + \mu)(q - p), \quad k_1 = 1, \quad k_2 = p - q - \lambda N, \quad k_3 = (\lambda - \mu)N.$$

We have a functor $F_{n,p,\mu}^\lambda$ from the category of $D^\lambda(\mathrm{GL}(N)/\mathrm{GL}(p) \times \mathrm{GL}(q))$ -modules to the category of representations of the type BC_n dDAHA with such parameters.

10.3. The trigonometric degeneration of the DAHA. In [Ch], Cherednik defined the dDAHA of a root system as a suitable quasi-classical limit of the DAHA. In this section, we explain how to apply this procedure to the DAHA of type $C^\vee C_n$ to recover the presentation of the dDAHA in Section 10.2. Thus we take $\mathcal{K} = \mathbb{C}((\hbar))$ in the definitions of Section 2.2

Recall that in [S], we have a faithful representation of the DAHA of type $C^\vee C_n$ which is given by follows. Let $\mathbb{C}[x] = \mathbb{C}[x_1^\pm, \dots, x_n^\pm]$, with the BC_n Weyl group acting by permuting and inverting the x_i . Define

$$\begin{aligned} \pi(X_i) &:= x_i, \\ \pi(T_0) &:= t_0 + t_0^{-1} \frac{(1 - vt_0 u_0 x_1^{-1})(1 + vt_0 u_0^{-1} x_1^{-1})}{1 - v^2 x_1^{-2}} (s_0 - 1), \\ \pi(T_i) &:= t + t^{-1} \frac{1 - t^2 x_i x_{i+1}^{-1}}{1 - x_i x_{i+1}^{-1}} (s_i - 1), \\ \pi(T_n) &:= t_n + t_n^{-1} \frac{(1 - t_n u_n x_n)(1 + t_n u_n^{-1} x_n)}{1 - x_n^2} (\gamma_n - 1), \end{aligned}$$

for, $i = 1, \dots, n - 1$. Then we have

Theorem 10.3 ([S], Theorem 3.1, 3.2). *The map π extends to a faithful representation of the $C^\vee C_n$ DAHA on $\mathbb{C}[x]$.*

Let $m_1, \dots, m_6 \in \mathbb{C}$, and define the following elements of $\mathbb{C}[[\hbar]]$:

$$q = e^{\hbar}, \quad t = q^{m_1}, \quad t_n = q^{m_2}, \quad t_0 = q^{m_3}, \quad u_0 = q^{m_4}, \quad u_n = q^{m_5}, \quad v = q^{m_6}.$$

Let \mathcal{H}_\hbar denote the closed subalgebra of $\mathrm{End}_{\mathbb{C}[[\hbar]]}(\mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}][[\hbar]])$ generated by the operators in Theorem 10.3. As the formulas expressing X_i, T_0, T_i and T_n in terms of the x_i, s_0, s_i , and s_n are invertible in $\mathbb{C}[[\hbar]]$, \mathcal{H}_\hbar is also generated by the latter set of elements.

Proposition 10.4. *The natural map on the (lower-case) generators induces an isomorphism $\mathcal{H}_\hbar/\hbar\mathcal{H}_\hbar \cong \mathcal{H}^{\mathrm{deg}}(t, k_1, k_2, k_3)$.*

Proof. By a direct computation, which we omit, it can be seen that the relations of the $C^\vee C_n$ type DAHA degenerate to the relations in the type BC_n degenerate double affine Hecke algebra. The parameter correspondence is given by

$$(34) \quad k_1 = m_1, \quad k_2 = m_2, \quad k_3 = m_3 = m_4 + m_5, \quad t = m_2 + m_3 + m_6.$$

□

10.4. The trigonometric degeneration of \mathbf{B}_σ . In this subsection, we let $\sigma \in \mathbb{C}$, and define the power series

$$\mathbf{q} := e^{\hbar}, \quad \mathbf{q}^\sigma := e^{\hbar\sigma} \in \mathbb{C}[[\hbar]].$$

In this way the algebras $\mathcal{U}_{\mathbf{q}}(\mathfrak{g})$ and \mathbf{B}_σ considered throughout become $\mathbb{C}[[\hbar]]$ -algebras.

Recall that a $\mathbb{C}[[\hbar]]$ -subalgebra \mathbf{B} of a $\mathbb{C}[[\hbar]]$ -algebra \mathbf{A} is called *saturated* if $\hbar a \in \mathbf{B} \Rightarrow a \in \mathbf{B}$. The saturation \mathbf{B}^s of \mathbf{B} is the smallest saturated subalgebra containing \mathbf{B} . The quasi-classical limit of a saturated subalgebra $\mathbf{B} \subset \mathbf{A}$ is the subalgebra $\mathbf{B}/\hbar\mathbf{B}$ of $\mathbf{A}/\hbar\mathbf{A}$. The following is an elaboration of [DS], Remark 6.4:

Claim 10.5. *For all $\sigma \in \mathbb{C}$, the quasi-classical limit of the subalgebra \mathbf{B}_σ^s is $\mathcal{U}(\mathfrak{k}')$, where \mathfrak{k}' is the subalgebra of \mathfrak{gl}_N defined in Section 7.4.*

Proof. As remarked in the proof of Proposition 7.6, the relations of the reflection equation algebra imply that \mathbf{B}_σ is spanned over $\mathbb{C}[[\hbar]]$ by ordered monomials in the c_{il} , and therefore its saturation \mathbf{B}_σ^s is a saturated subalgebra whose quasi-classical limit is generated by the quasi-classical limits of the generators c_{il} . Thus it remains only to compute the quasi-classical limits of the c_{il} and check that they coincide with the generators of $\mathcal{U}(\mathfrak{k}')$.

We recall the formula for the generators c_{il} :

$$c_{il} = \sum_{j,k=1}^m l_{ij}^+(J_V)_{jk} S(l_{kl}^-).$$

The classical limits of each l_{ij}^\pm are δ_{ij} . We recall the well-known formulas for the quasi-classical limits of the l_{ij}^\pm :

$$\begin{aligned} \lim_{\mathbf{q} \rightarrow 1} \frac{l_{ij}^\pm}{\mathbf{q} - \mathbf{q}^{-1}} &= - \lim_{\mathbf{q} \rightarrow 1} \frac{S(l_{ij}^\pm)}{\mathbf{q} - \mathbf{q}^{-1}} = \pm E_j^i, \text{ for } i \neq j; \\ \lim_{\mathbf{q} \rightarrow 1} \frac{2(l_{ii}^+ - l_{jj}^-)}{\mathbf{q} - \mathbf{q}^{-1}} &= E_i^i + E_j^j. \end{aligned}$$

The only terms in the summation expression for c_{il} which will contribute to the quasi-classical limit are those in which either $i = j$ or $k = l$; in all other cases, the term will vanish to second order in \hbar , and thus its quasiclassical limit will be zero. We have six cases to compute, according to the block form of J^σ .

Case 1a: $1 \leq i < l \leq p$.

$$\begin{aligned} \lim_{\mathbf{q} \rightarrow 1} \frac{c_{il}}{\mathbf{q} - \mathbf{q}^{-1}} &= \lim_{\mathbf{q} \rightarrow 1} \frac{l_{i,N-l+1}^+ S(l_{ll}^-) + l_{ii}^+ S(l_{N-i+1,l}^-)}{\mathbf{q} - \mathbf{q}^{-1}} \\ &= E_{N-l+1}^i + E_l^{N-i+1}; \end{aligned}$$

Case 1b: $1 \leq l < i \leq p$.

$$\begin{aligned} \lim_{q \rightarrow 1} \frac{c_{il}}{q - q^{-1}} &= \lim_{q \rightarrow 1} \frac{l_{i,i}^+ S(l_{N-i+1,l}^-) + l_{i,N-l+1}^+ S(l_{l,l}^-)}{q - q^{-1}} \\ &= E_l^{N-i+1} + E_{N-l+1}^i; \end{aligned}$$

Case 1c: $1 \leq i = l \leq p$.

$$\begin{aligned} \lim_{q \rightarrow 1} \frac{c_{ii}}{q - q^{-1}} &= \lim_{q \rightarrow 1} \frac{l_{ii}^+ S(l_{ii}^-)(q^\sigma - q^{-\sigma}) + l_{i,i}^+ S(l_{N-i+1,i}^-) + l_{i,N-i+1}^+ S(l_{i,i}^-)}{q - q^{-1}} \\ &= \sigma + E_i^{N-i+1} + E_{N-i+1}^i; \end{aligned}$$

Case 2: $1 \leq i \leq p, p+1 \leq l \leq N-p$.

$$\begin{aligned} \lim_{q \rightarrow 1} \frac{c_{il}}{q - q^{-1}} &= \lim_{q \rightarrow 1} \frac{l_{i,i}^+ S(l_{N-i+1,l}^-) - q^{-\sigma} l_{i,l}^+ S(l_{l,l}^-)}{q - q^{-1}} \\ &= E_l^{N-i+1} - E_l^i; \end{aligned}$$

Case 3a: $N-p+1 \leq l \leq N, 1 \leq i < N-l+1$.

$$\begin{aligned} \lim_{q \rightarrow 1} \frac{c_{il}}{q - q^{-1}} &= \lim_{q \rightarrow 1} \frac{l_{i,i}^+ S(l_{N-i+1,l}^-) + l_{i,N-l+1}^+ S(l_{l,l}^-)}{q - q^{-1}} \\ &= E_l^{N-i+1} + E_{N-l+1}^i; \end{aligned}$$

Case 3b: $N-p+1 \leq l \leq N, i = N-l+1$.

$$\begin{aligned} \lim_{q \rightarrow 1} \frac{2 - 2c_{il}}{q - q^{-1}} &= \lim_{q \rightarrow 1} \frac{2(1 - l_{i,i}^+ S(l_{N-i+1,N-i+1}^-))}{q - q^{-1}} \\ &= \lim_{q \rightarrow 1} \frac{2(l_{N-i+1,N-i+1}^- - l_{i,i}^+) S(l_{N-i+1,N-i+1}^-)}{q - q^{-1}} \\ &= -E_{N-i+1}^{N-i+1} - E_i^i; \end{aligned}$$

Case 4: $1 \leq l \leq p, p+1 \leq i \leq N-p$.

$$\begin{aligned} \lim_{q \rightarrow 1} \frac{c_{il}}{q - q^{-1}} &= \lim_{q \rightarrow 1} \frac{-q^{-\sigma} l_{i,i}^+ S(l_{i,l}^-) + l_{i,N-l+1}^+ S(l_{l,l}^-)}{q - q^{-1}} \\ &= E_{N-l+1}^i - E_l^i; \end{aligned}$$

Case 5a: $p+1 \leq i < l \leq N-p$.

$$\lim_{q \rightarrow 1} \frac{c_{il}}{q - q^{-1}} = \lim_{q \rightarrow 1} \frac{q^{-\sigma} l_{i,l}^+ S(l_{l,l}^-)}{q - q^{-1}} = E_l^i;$$

Case 5b: $p+1 \leq i = l \leq N-p$.

$$\begin{aligned} \lim_{q \rightarrow 1} \frac{q^{-\sigma} + c_{ii}}{q - q^{-1}} &= \lim_{q \rightarrow 1} \frac{q^{-\sigma} - q^{-\sigma} l_{i,i}^+ S(l_{i,i}^-)}{q - q^{-1}} \\ &= \lim_{q \rightarrow 1} \frac{q^{-\sigma} (l_{i,i}^- - l_{i,i}^+) S(l_{i,i}^-)}{q - q^{-1}} \\ &= -E_i^i; \end{aligned}$$

Case 5c: $p+1 \leq l < i \leq N-p$.

$$\lim_{q \rightarrow 1} \frac{c_{il}}{q - q^{-1}} = \lim_{q \rightarrow 1} \frac{-q^{-\sigma} l_{i,i}^+ S(l_{i,l}^-)}{q - q^{-1}} = -E_l^i;$$

Case 6a: $N-p+1 \leq i \leq N$, $1 \leq l < N-i+1$.

$$\begin{aligned} \lim_{q \rightarrow 1} \frac{c_{il}}{q - q^{-1}} &= \lim_{q \rightarrow 1} \frac{l_{i,i}^+ S(l_{N-i+1,l}^-) + l_{i,N-l+1}^+ S(l_{l,l}^-)}{q - q^{-1}} \\ &= E_l^{N-i+1} + E_{N-l+1}^i; \end{aligned}$$

Case 6b: $N-p+1 \leq i \leq N$, $l = N-i+1$.

$$\begin{aligned} \lim_{q \rightarrow 1} \frac{2 - 2c_{il}}{q - q^{-1}} &= \lim_{q \rightarrow 1} \frac{2 \left(1 - l_{i,i}^+ S(l_{N-i+1,N-i+1}^-) \right)}{q - q^{-1}} \\ &= \lim_{q \rightarrow 1} \frac{2(l_{N-i+1,N-i+1}^- - l_{i,i}^+) S(l_{N-i+1,N-i+1}^-)}{q - q^{-1}} \\ &= -E_i^i - E_{N-i+1}^{N-i+1}. \end{aligned}$$

Finally, we let

$$(35) \quad g = \sum_{k=1}^p E_k^k - \sum_{k=p+1}^n E_k^k + \sum_{k=1}^p E_{n-k+1}^k + \sum_{k=1}^p E_k^{n-k+1}$$

and conjugate each of the above elements by g . We have

$$\begin{aligned} g(E_{N-l+1}^i + E_l^{N-i+1})g^{-1} &= E_l^i - E_{N-l+1}^{N-i+1}, \text{ in Case 1a;} \\ g(E_l^{N-i+1} + E_{N-l+1}^i)g^{-1} &= E_l^i - E_{N-l+1}^{N-i+1}, \text{ in Case 1b;} \\ \sigma + g(E_i^{N-i+1} + E_{N-i+1}^i)g^{-1} &= \sigma + E_i^i - E_{N-i+1}^{N-i+1}, \text{ in Case 1c;} \\ g(E_l^{N-i+1} - E_l^i)g^{-1} &= E_l^{N-i+1}, \text{ in Case 2;} \\ g(E_l^{N-i+1} + E_{N-l+1}^i)g^{-1} &= E_l^{N-i+1} + E_{N-l+1}^i, \text{ in Case 3a;} \\ g(-E_i^i - E_{N-i+1}^{N-i+1})g^{-1} &= -E_i^i - E_{N-i+1}^{N-i+1}, \text{ for Cases 3b and 6b;} \\ g(E_{N-l+1}^i - E_l^i)g^{-1} &= 2E_{N-l+1}^i, \text{ in Case 4;} \\ g(E_l^i)g^{-1} &= E_l^i, \text{ in Cases 5a, b and c;} \\ g(E_l^{N-i+1} + E_{N-l+1}^i)g^{-1} &= E_l^{N-i+1} + E_{N-l+1}^i, \text{ in Case 6a;} \end{aligned}$$

Thus we see by direct inspection that the quasi-classical limit of the subalgebra \mathbf{B}_σ is the algebra $\mathcal{U}(\mathfrak{k}')$. \square

10.5. The trigonometric degeneration of the character χ_τ^η . By trigonometric degeneration of a character $\chi : \mathbf{B}_\sigma \rightarrow \mathbb{C}$ we will mean the following: first we work over $\mathbb{C}[[\hbar]]$, and set $q = e^\hbar$, $q^\sigma = e^{\hbar\sigma}$. We thus view χ as a homomorphism to $\mathbb{C}[[\hbar]]$ instead. We send $a \in \mathbf{B}_\sigma^s / \hbar \mathbf{B}_\sigma^s$ to $\chi(\hat{a}) \bmod \hbar$ for any lift \hat{a} of a .

We now apply the explicit computations above to compute the trigonometric degeneration of the characters χ_τ^η . In order to be compatible with the conventions of [EFM], we will consider the character $\tilde{\chi}_\tau^\eta : \mathfrak{gl}_p \times \mathfrak{gl}_q \rightarrow \mathfrak{k}' \rightarrow \mathbb{C}$, obtained by precomposing with conjugation by g^{-1} , and applying the quasi-classical limit of the character $\chi_\tau^\eta : \mathbf{B}_\sigma^s \rightarrow \mathbb{C}$. We compute that:

$$\tilde{\chi}_\tau^\eta \left(\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \right) = \frac{\eta + \tau - \sigma}{2} \text{tr } A_1 + \frac{\eta + \sigma - \tau}{2} \text{tr } A_2.$$

Thus, we have that

$$\tilde{\chi}_\tau^\eta = \left(\frac{\eta}{2} + \frac{(p-q)(\tau-\sigma)}{2N} \right) \text{tr} + \frac{(\tau-\sigma)}{N} \chi,$$

where χ is that from equation (31).

Similarly, we can compute the character $\tilde{\lambda}_\nu^\omega : \mathfrak{gl}_p \times \mathfrak{gl}_q \rightarrow \mathbb{C}$ obtained from λ_ν^ω by quasi-classical limit. We have

$$\tilde{\lambda}_\nu^\omega = \left(\frac{\omega}{2} + \frac{(p-q)(\rho-\nu)}{2N} \right) \text{tr} + \frac{(\rho-\nu)}{N} \chi.$$

10.6. An alternate presentation for the DAHA. In this section, we recall an alternate presentation for the DAHA (e.g. [S],[EGO]), and prove that it coincides with Definition 2.3.

Let $[a, b]$ denote the set of integers between a and b inclusive, regardless of which is larger. Recall the elements $T_{(i \dots j)}$ and P_i from Section 2. By direct computation, we have the following:

Lemma 10.6. *We have the following relations:*

$$T_{(i \dots j)} T_{(k \dots l)} = \begin{cases} T_{(k \dots l)} T_{(i \dots j)}, & [i, j] \cap [k, l] = \emptyset, \\ T_{(k \dots l)} T_{(i+1 \dots j+1)}, & [i, j] \subsetneq [k, l], k > l, \\ T_{(k \dots l)} T_{(i-1 \dots j-1)}, & [i, j] \subsetneq [k, l], k < l, \end{cases}$$

$$T_i P_{i+1} T_i = P_i, \quad T_i P_j = P_j T_i \quad (j \neq i, i+1),$$

$$P_i P_j = P_j P_i, \quad i, j = 1, \dots, n-1.$$

Consider the following elements:

$$(36) \quad Y_i := P_i T_{(i \dots 1)} T_0 T_{(i \dots 1)}^{-1},$$

$$(37) \quad X_i := P_i^{-1} T_{(1 \dots i)}^{-1} K_0^{-1} T_{(1 \dots i)}.$$

Proposition 10.7. *$\tilde{\mathcal{B}}_n$ is generated by the group \mathcal{B}_n and elements $X_1, \dots, X_n, Y_1, \dots, Y_n$, with the relations:*

$$T_i Y_{i+1} T_i = Y_i, \quad T_i X_i T_i = X_{i+1}, \quad X_i X_j = X_j X_i, \quad Y_i Y_j = Y_j Y_i \quad (i, j = 1, \dots, n),$$

$$T_i Y_j = Y_j T_i, \quad T_i X_j = X_j T_i \quad (j \neq i, i+1), \quad T_n Y_{n-1} = Y_{n-1} T_n, \quad T_n X_{n-1} = X_{n-1} T_n,$$

$$X_i (P_1^{-1} Y_1) = (P_1^{-1} Y_1) X_i \quad (i = 2, \dots, n-1).$$

Proof. Let $\tilde{\mathcal{B}}'$ denote the group specified in the proposition, and reserve $\tilde{\mathcal{B}}$ for the group given by Definition 2.3. We define $\phi : \tilde{\mathcal{B}}' \rightarrow \tilde{\mathcal{B}}$ on generators:

$$\phi : T_i \mapsto T_i, \quad i = 1, \dots, n,$$

$$X_i \mapsto P_i^{-1} T_{(1 \dots i)}^{-1} K_0^{-1} T_{(1 \dots i)}, \quad i = 1, \dots, n,$$

$$Y_i \mapsto P_i T_{(i \dots 1)} T_0 T_{(i \dots 1)}^{-1}, \quad i = 1, \dots, n.$$

We leave it to the reader to verify that ϕ defines an isomorphism. \square

Remark 10.8. Along the lines of Remark 2.5, the isomorphism ϕ admits the following geometric interpretation: every elliptic curve $E = \mathbb{C}/\Lambda$ admits a \mathbb{Z}_2 action, $z \mapsto -z$. Let E° denote the complement of the fixed points. It is easy to see that E°/\mathbb{Z}_2 is homeomorphic to $\mathbb{P}^1 \setminus \{p_1, p_2, p_3, p_4\}$. The generators X_i and Y_j of $\tilde{\mathcal{B}}_n$ correspond to the horizontal and vertical cycles on E , as in [J], Figure 1. The generators T_0 , T_n , and K_0 correspond to loops around p_1, p_2 , and p_3 , respectively, so that $(K_0 P_1 T_0)^{-1}$ corresponds to a loop around p_4 .

Corollary 10.9. *The double affine Hecke algebra is a quotient of $\mathcal{K}[\tilde{\mathcal{B}}]$ by the relations:*

$$Y_n T_n^{-1} \sim t_0, \quad T_n \sim t_n, \quad X_n^{-1} T_n^{-1} \sim u_n,$$

$$v^{-1} Y_1^{-1} P_1 X_1 \sim u_0, \quad T_i \sim t \quad (i = 1, \dots, n-1).$$

Remark 10.10. The operators T_0 defined in Section 5.2 determine operators Y_i , via the isomorphism asserted in Proposition 10.7. It should be noted that these coincide with the inverse of the operators Y_i which appeared in [J] for the A_{n-1} construction, except that those involved \mathfrak{sl}_N , rather than \mathfrak{gl}_N .

10.7. The quasi-classical limit of Theorems 9.1 and 8.1. In this section, we compute the quasi-classical limits of the operators appearing in Theorems 9.1 and 8.1, making use of the alternate presentation for the $C^\vee C_n$ DAHA from the previous section. By comparing the results with the operators in [EFM], we can give a reproof of Theorems 10.1 and 10.2. This serves as a consistency check for both papers.

It is well known that the quasi-classical limit of the R -matrix of $\mathcal{U}_q(\mathfrak{gl}_N)$ is

$$1 + \hbar r \mod \hbar^2,$$

where r denotes the classical R -matrix for \mathfrak{gl}_N . Thus, for $i = 1, \dots, n-1$, the quasi-classical limit of T_i is

$$s_i(1 + \hbar r_{i,i+1}) \mod \hbar^2.$$

By direct computation, the classical limit of T_n is

$$J' + \hbar \sigma \hat{J} \mod \hbar^2,$$

where $\hat{J} = 2 \sum_{i \leq p} E_i^i + \sum_{p+1 \leq i \leq q} E_i^i$, and J' is the classical matrix from equation (25).

Lemma 10.11. *When $\mathbf{U} = \mathcal{U}(\mathfrak{gl}_N)$, the operator K_0 acts as $(AJA^{-1})_j^i \otimes E_i^j$.*

Proof. The proof is by direct computation in the symmetric category $\mathcal{U}(\mathfrak{g})\text{-mod}$, and relies on the triviality of the braiding to simplify K_0 . We may choose a basis diagonalizing J , and rewrite equation (19) in coordinates, ignoring appearance of R -matrices, identifying ${}^*V \cong V^*$ canonically, and noting that the classical limit (in

this basis) of J^σ is J :

$$\begin{aligned}
 K_0 &= \sum c_{Jv_k \otimes v^k, v_j \otimes v^i} \otimes E_i^j \\
 &= \sum c_{v^k, v_j} c_{Jv_k \otimes v^i} \otimes E_i^j \\
 &= \sum c_{v^k, v_j} J_k^l S(c_{v^i, v_l}) \otimes E_i^j \\
 &= \sum a_j^k J_k^l S(a_l^i) \otimes E_i^j \\
 &= \sum (AJA^{-1})_{ji} \otimes E_i^j.
 \end{aligned}$$

□

Proposition 10.12. *The classical limit of X_1 is $\sum (AJA^{-1}J)_j^i \otimes E_i^j$*

Proof. We have $X_1 = P_1^{-1}K_0^{-1}$. The classical limit of P_1^{-1} is J_1 , by direct computation, using triviality of the braiding, and the fact that $J = J^{-1}$. Thus, by the lemma, we have:

$$X_1 = \sum (AJA^{-1})_j^i \otimes J_{kl} E_k^l E_i^j = \sum (AJA^{-1}J)_j^k \otimes E_k^j,$$

as desired. □

Define $\hat{y}_i \in \text{End}_{\mathbb{C}}(M \otimes V^{\otimes n})$ by the equation $Y_i \equiv 1 + \hbar \hat{y}_i \pmod{\hbar^2}$. As noted in Remark 10.10, the operators Y_i determined by our choice for T_0 and Proposition 10.7 coincide with the inverse of those of [J]. In order to prove Theorem 8.1, we rescaled T_0 and thus Y_1 by $q^{\eta-N}$ and thus the quasi-classical limit of y_1 is computed by:

Proposition 10.13 (see [J], Proposition 6.14). *The operator \hat{y}_1 is given by:*¹⁰

$$\hat{y}_i = -\Omega_{0i} - \sum_{j < i} s_{ij} + \frac{\eta - N}{2},$$

where is the $\Omega = \sum_{i,j} E_i^j \otimes E_j^i \in \text{Sym}^2(\mathfrak{g})^{\mathfrak{g}}$ is the Casimir element for $\mathfrak{g} = \mathfrak{gl}_N$.

The following proposition allows us to compare \hat{y}_i with the operators y_i from Section 10.1. We have:

Proposition 10.14. *As an operator on the $(\mathfrak{k}, \tilde{\chi}_\tau^\eta)$ -invariants, we have*

$$y_1 = -\Omega_{01} + \frac{\eta - N}{2} + \frac{(\tau - \sigma) - \mu N}{2} \gamma_1.$$

¹⁰in that construction, $t = q^k$ is the parameter for the quantum group $\mathcal{U}_t(\mathfrak{sl}_N)$, and thus the factor k multiplies \tilde{y}_i . Also, since we work with \mathfrak{gl}_N , there is not the shift $\frac{i-1}{N}$, which occurs in Proposition 6.14 of [J], because $\Omega^{\mathfrak{sl}_N} = \Omega^{\mathfrak{gl}_N} - \frac{1}{N} \text{id}_N \otimes \text{id}_N$

Proof. Recall the summation convention $\sum_{ij} := \sum_{i,j=1}^p + \sum_{i,j=p+1}^N$ from [EFM]. First, we set $i = 1$ in equation (32), and simplify the summations over k :

$$\begin{aligned}
S &= \frac{1}{2} \sum_{k>1} s_{1k} + \frac{1}{2} \sum_{k>1} s_{1k} \gamma_1 \gamma_k \\
&= \frac{1}{2} \sum_{k>1} \sum_{i,j} (E_i^j)_1 \otimes (E_j^i)_k + \frac{1}{2} \sum_{k>1} \sum_{i,j} (E_i^j J)_1 \otimes (E_j^i J)_k \\
&= \sum_{k>1} \sum_{i,j} (E_i^j)_1 \otimes (E_j^i)_k \\
&\quad (\text{applying the } \tilde{\chi}_\tau^\eta\text{-invariant property, to the tensor factors in } \mathfrak{k}) \\
&= \sum_{i,j} (E_i^j)_1 \tilde{\chi}_\tau^\eta(E_j^i) - \sum_{i,j} (E_i^j)_0 \otimes (E_j^i)_1 - p \sum_{i \leq p} (E_i^i)_1 - q \sum_{i > p} (E_i^i)_1 \\
&= \frac{\eta}{2} + \frac{\tau - \sigma}{2} \left(\sum_{i \leq p} (E_i^i)_1 - \sum_{i > p} (E_i^i)_1 \right) - \sum_{i,j} (E_j^i)_0 \otimes (E_j^i)_1 \\
&\quad - p \sum_{i \leq p} (E_i^i)_1 - q \sum_{i > p} (E_i^i)_1.
\end{aligned}$$

Thus, we may rewrite equation (32):

$$y_1 = - \sum_{i,j} (E_i^j)_0 \otimes (E_j^i)_1 + \frac{\eta - N}{2} + \frac{(\tau - \sigma) - \mu N}{2} \gamma_1.$$

□

Finally, let:

$$\begin{aligned}
\sigma &= p - q - \lambda N \\
\tau &= (\mu - \lambda)N + p - q \\
\nu - \rho &= (\lambda - \mu)N \\
\eta - \omega &= N + \frac{2n}{N} + \lambda(q - p) - 2\mu p
\end{aligned}$$

Comparing with (34), we see that k_1, k_2, k_3 and t from the degeneration of the DAHA agree with the parameters of Theorem 10.2. On the other hand, we have shown that the coideal subalgebras \mathbf{B}_σ and \mathbf{B}'_ρ both degenerate to the subalgebra $\mathcal{U}(\mathfrak{gl}_p \times \mathfrak{gl}_q)$, while the characters $\tilde{\chi}_\tau^\eta$ and $\tilde{\lambda}_\rho^\rho$ degenerate to the characters $\mu\chi$ and $(\mu - \lambda)\chi$, respectively, upon restriction to $\mathfrak{gl}_p \times \mathfrak{gl}_q$.

Thus we may recover Theorems 10.1 and 10.2 as follows. η records the spectrum of the center of \mathfrak{gl}_N on M , which is discarded in [EFM], who consider instead \mathfrak{sl}_N . Thus by summing the $F_n^{\sigma,\eta,\tau}(M)$ over all η , and $F_{n,\rho,\omega,\nu}^{\sigma,\eta,\tau}(M)$ over all η and ω , we recover the spaces of Theorems 10.1 and 10.2¹¹, respectively as quasi-classical limits. We have shown that the operators X_i and T_j degenerate to x_i and s_j , respectively, for $i, j = 1, \dots, n$, and we have shown that $\hat{y}_i = y_i$. Thus the entire constructions of [EFM] are recovered as quasi-classical limits of the present results.

¹¹In that paper, the authors consider $\lambda\chi$ -twisted D -modules, and μ -invariants. This coincides with $\lambda\chi$ -ad-invariants, and $\mu\chi$ left-invariants, or equivalently $(\mu - \lambda)\chi$ right-invariants and $\mu\chi$ left-invariants.

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